

# SPHERICALLY AVERAGED MAXIMAL FUNCTION AND SCATTERING FOR THE 2D CUBIC DERIVATIVE SCHRÖDINGER EQUATION

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ABSTRACT. We prove scattering for the 2D cubic derivative Schrödinger equation with small data in the critical Besov space with one degree angular regularity. The main new ingredient is that we prove a spherically averaged maximal function estimate for the 2D Schrödinger equation. We also prove a global well-posedness result for the 2D Schrödinger map in the critical Besov space with one degree angular regularity. The key ingredients for the latter results are the spherically averaged maximal function estimate, null form structure observed in [2], as well as the generalised spherically averaged Strichartz estimates obtained in [11] in order to exploit the null form structure.

## CONTENTS

1. Introduction	1
2. Definitions and Notations	4
3. Spherically averaged maximal function estimates	6
4. Cubic Derivative NLS	12
5. Schrödinger map in two dimensions	15
Acknowledgment	25
References	25

## 1. INTRODUCTION

In this paper, we study the Cauchy problem for the cubic derivative Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 \mathbf{a} \cdot \nabla u + u^2 \mathbf{b} \cdot \nabla \bar{u}, \\ u(x, 0) = u_0(x) \end{cases} \quad (1.1)$$

where  $u(x, t) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{C}$ ,  $\mathbf{a}, \mathbf{b} \in \mathbb{C}^2$ . The equation (1.1) arises from the strongly interacting many-body systems near criticality as recently described in terms of nonlinear dynamics [8]. The Schrödinger equation with derivative in the nonlinearity of the form

$$i\partial_t u + \Delta u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad (1.2)$$

has been studied extensively, e.g. see the introduction of [16, 24] for the history of the study. Besides equation (1.1) and the well-known one-dimension derivative Schrödinger equation, (1.2) contains another important model known as the

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Schrödinger maps

$$\partial_t s = s \times \Delta_x s, \quad s(0) = s_0, \quad (1.3)$$

where  $s : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ . It was known that under stereographic projection (see Section 5 below) (1.3) is equivalent to

$$i\partial_t u + \Delta u = \frac{2\bar{u}}{1+|u|^2} \sum_{i=1}^n (\partial_{x_i} u)^2, \quad (1.4)$$

and we see that the cubic term  $\bar{u}(\partial_{x_i} u)^2$  serves as the first term in the Taylor expansion of the above nonlinear term.

Note that the equation (1.1) is invariant under the following scaling transform: for  $\lambda > 0$

$$u(x, t) \rightarrow \lambda^{1/2} u(\lambda x, \lambda^2 t), \quad u_0(x) \rightarrow \lambda^{1/2} u_0(\lambda x),$$

then we see that the critical space for (1.1) is  $\dot{H}^{\frac{n-1}{2}}$  in the sense of scaling. Because of the loss of the derivative, the usual Strichartz analysis as for the power type nonlinearity doesn't work for (1.1). One needs some estimates with stronger smoothing effect. Kenig, Ponce, and Vega [16] introduced for the first time a method to obtain local well-posedness for general derivative Schrödinger equations. This method combines “local smoothing estimates”, “inhomogeneous local smoothing estimates”, which give the crucial gain of one derivative, and “maximal function estimates”. In the study of Schrödinger map, Ionescu and Kenig [12, 13] introduced the anisotropic local smoothing and maximal function estimates for Schrödinger equation. It was proved in [13] that the following local smoothing estimates hold

$$\begin{aligned} \|e^{it\Delta} P_{k, \mathbf{e}_1} f\|_{L_{x_1}^\infty L_{\bar{x}, t}^2} &\lesssim 2^{-k/2} \|f\|_2, \\ \left\| \int_0^t e^{i(t-s)\Delta} P_{k, \mathbf{e}_1} g \right\|_{L_{x_1}^\infty L_{\bar{x}, t}^2} &\lesssim 2^{-k} \|g\|_{L_{x_1}^1 L_{\bar{x}, t}^2}, \end{aligned}$$

where  $P_{k, \mathbf{e}_1} f = \mathcal{F}^{-1} 1_{|\xi_1| \sim 2^k} 1_{|\xi| \sim 2^k} \hat{f}$  (roughly, see Section 2 for the definition). In order to apply these estimates to deal with the cubic nonlinear terms, the following maximal function appears naturally

$$\|e^{it\Delta} P_k f\|_{L_{x_1}^2 L_{\bar{x}, t}^\infty} \lesssim 2^{(n-1)k/2} \|f\|_2. \quad (1.5)$$

It was proved in [13] that (1.5) holds if  $n \geq 3$ . These estimates played key roles in the consequent study of Schrödinger map, e.g. in [4]. For (1.1), in three dimensions and higher, one could gather these estimates to obtain global well-posedness and scattering in the critical Besov space, see [26] which also generalized the estimates and results to the non-elliptic case. In [24, 25] these estimates were generalized to the modulation space and sharp global well-posedness in modulation spaces for (1.1) (also in the non-elliptic case) with  $n \geq 3$  were obtained.

However, if  $n = 2$ , (1.5) fails. Thus the cubic nonlinear term in two dimensions is more difficult. To the author's knowledge, there are two approaches to deal with this difficulty. The first one was developed in [4] which uses the Galilean invariance of the Schrödinger propagator. The space  $L_{x_1}^2 L_{x_2, t}^\infty$  is replaced with a sum of Galilean transforms of it. The idea of using such sums of spaces as substitutes is due to Tataru [21]. The space is defined for any finite time interval  $[-T, T]$ , but the estimates in [4] are independent of  $T$ . The second one was developed in [24] which proves the

following estimate via the Gabor frame representation of linear Schrödinger solution: for  $1 \leq r < 2$

$$\|e^{it\Delta}P_0f\|_{L_{x_1}^2L_{\bar{x},t}^\infty} \lesssim \|f\|_r.$$

Then with this well-posedness and scattering for (1.1) with  $n = 2$  were proved for suitable data in some modulation space.

In this paper, we take another approach. Our ideas are inspired by [22] and the recent work [10]. First, since (1.5) only fails “logarithmically” for  $n = 2$ , we find that the spherically averaged maximal function estimate holds. This is like the spherically averaged endpoint Strichartz estimate for the 2D Schrödinger equation that was studied in [22]. Note that, the Strichartz space  $L_t^2L_x^\infty$  is rotational invariant, however the anisotropic space  $L_{x_1}^2L_{\bar{x},t}^\infty$  is not. It is a bit surprising that we have

**Theorem 1.1.** *There exists  $C > 0$  such that for  $k \in \mathbb{Z}$ ,  $u_0 \in L^2(\mathbb{R}^2)$ , one has*

$$\|e^{it\Delta}P_ku_0\|_{L_{x_1}^2L_{x_2,t}^\infty L_\theta^2} \leq C2^{k/2}\|u_0\|_2. \quad (1.6)$$

See Section 2 for the definition of the space  $L_{x_1}^2L_{x_2,t}^\infty L_\theta^2$  and  $P_k$ . Then we use an argument of [26] (which is in the spirit of [4]) to derive the corresponding inhomogeneous estimate. To use this norm to the equation (1.1), as in [10] we assume sufficient regularity on the sphere variable such that the space on the sphere is an algebra. Not like the Strichartz space, the local smoothing/maximal function space is anisotropic in  $x$  which makes it not very compatible with the spherical average. For example, we do not have compare between  $L_{x_1}^2L_{x_2,t}^\infty L_\theta^2$  and  $L_{x_1}^2L_{x_2,t}^\infty$ . Fortunately, we can still close the iteration arguments in these spaces. We show

**Theorem 1.2.** *Assume  $n = 2$ ,  $u_0 \in \dot{B}_{2,1,\theta}^{1/2,1}$  with  $\|u_0\|_{\dot{B}_{2,1,\theta}^{1/2,1}} = \varepsilon_0 \ll 1$ . Then there exists a unique global solution  $u$  to (1.1) such that  $\|u\|_{F^{1/2}} \lesssim \varepsilon_0$ . Moreover, the map  $u_0 \rightarrow u$  is Lipschitz from  $\dot{B}_{2,1,\theta}^{1/2,1}$  to  $C(\mathbb{R}; \dot{B}_{2,1,\theta}^{1/2,1})$ , and scattering holds in this space.*

*Remark 1.* In Theorem 1.2,  $u_0 \in \dot{B}_{2,1,\theta}^{1/2,1}$  means that  $u_0 \in \dot{B}_{2,1}^{1/2}$  and its spherical derivative  $\partial_\theta u_0 \in \dot{B}_{2,1}^{1/2}$ , and  $\|u_0\|_{\dot{B}_{2,1,\theta}^{1/2,1}} = \|u_0\|_{\dot{B}_{2,1}^{1/2}} + \|\partial_\theta u_0\|_{\dot{B}_{2,1}^{1/2}}$ . We do not need  $X^{s,b}$ -structure for the proof of Theorem 1.2, and see Section 4 for the definition of  $F^s$ . Note that if  $u_0 \in \dot{B}_{2,1}^{1/2}$  is radial, then  $u_0 \in \dot{B}_{2,1,\theta}^{1/2,1}$ . This is a bit surprising that for radial data the problem is relatively simpler even though the radial symmetry is not preserved under the flow of (1.1).

Now we turn to the study of the Schrödinger map. It has also been studied extensively (also in the case in which the sphere  $\mathbb{S}^2$  is replaced by more general targets). Based on variants of the energy method, the local existence of the sufficiently smooth solutions were obtained, even for large data, see, for example, [19, 6, 9, 15] and the references therein. Similarly as (1.1), by the scaling we see the critical space for (1.3) is  $\dot{H}^{d/2}$ . Local well-posedness were obtained [12] for small data in  $H_Q^s(\mathbb{R}^n : \mathbb{S}^2)$ ,  $s > (n+1)/2$ . This was improved to  $s > n/2$  by Bejenaru [2]. Bejenaru observed for the first time in the setting of Schrödinger maps, that the gradient part of the nonlinearity in (1.4) has a certain null structure. Global well-posedness for small data in the critical Besov space in dimensions  $n \geq 3$  were obtained in [13], and independently in [3]. Recently, global well-posedness for small data in the critical Sobolev space were proved in [5] first for  $n \geq 4$ , and in [4] for  $n \geq 2$  where some

state of art techniques were built. We revisit the case  $n = 2$  using the new maximal function estimate. We prove

**Theorem 1.3.** *Assume  $n = 2$ , the Schrödinger map initial value problem (1.3) is globally well-posed for small data  $s_0 \in \dot{B}_Q^{1,1}(\mathbb{R}^2; \mathbb{S}^2)$ ,  $Q \in \mathbb{S}^2$ .*

*Remark 2.* The space  $\dot{B}_Q^{s,1}$  is defined by

$$\dot{B}_Q^{s,1} = \{f : \mathbb{R}^2 \rightarrow \mathbb{R}^3; f - Q \in \dot{B}_{2,1,\theta}^{s,1}, |f(x)| \equiv 1 \text{ a.e. in } \mathbb{R}^2\}.$$

In the proof of Theorem 1.3, we will use  $X^{s,b}$ -type space in order to exploit the null structure as in [2, 12]. In two dimension, there is a logarithmic problem to exploit the null structure which does not appear in 3D and higher. Fortunately, we can use the generalised spherically averaged Strichartz estimates obtained in [11] to overcome it. So the additional angular regularity is not only needed for the new maximal function, but also for using null structure.

## 2. DEFINITIONS AND NOTATIONS

For  $x, y \in \mathbb{R}$ ,  $x \lesssim y$  means that there exists a constant  $C$  such that  $x \leq Cy$ , and  $x \sim y$  means that  $x \lesssim y$  and  $y \lesssim x$ . We use  $\mathcal{F}(f)$ ,  $\hat{f}$  to denote the space-time Fourier transform of  $f$ , and  $\mathcal{F}_{x_i,t} f$  to denote the Fourier transform with respect to  $x_i, t$ .

Let  $\eta : \mathbb{R} \rightarrow [0, 1]$  be an even, non-negative, radially decreasing smooth function such that: a)  $\eta$  is compactly supported in  $\{\xi : |\xi| \leq 8/5\}$ ; b)  $\eta \equiv 1$  for  $|\xi| \leq 5/4$ . For  $k \in \mathbb{Z}$  let  $\chi_k(\xi) = \eta(\xi/2^k) - \eta(\xi/2^{k+1})$  and  $\chi_{\leq k}(\xi) = \eta(\xi/2^k)$ ,  $\tilde{\chi}_k(\xi) = \sum_{l=-9}^9 \chi_{k+l}(\xi)$  and then define the Littlewood-Paley projectors  $P_k, P_{\leq k}, P_{\geq k}$  on  $L^2(\mathbb{R}^2)$  by

$$\widehat{P_k u}(\xi) = \chi_k(|\xi|) \widehat{u}(\xi), \quad \widehat{P_{\leq k} u}(\xi) = \chi_{\leq k}(|\xi|) \widehat{u}(\xi),$$

and  $P_{\geq k} = I - P_{\leq k-1}$ . Let  $\mathbb{S}^1$  be the unit circle in  $\mathbb{R}^2$ . For  $\mathbf{e} \in \mathbb{S}^1$ , define  $\widehat{P_{k,\mathbf{e}} u}(\xi) = \tilde{\chi}_k(|\xi \cdot \mathbf{e}|) \chi_k(|\xi|) \widehat{u}(\xi)$ . Since for  $|\xi| \sim 2^k$  we have  $\sum_{l=-5}^5 \chi_{k+l}(\xi_1) + \sum_{l=-5}^5 \chi_{k+l}(\xi_2) \sim 1$ , then let

$$\beta_k^j(\xi) = \frac{\sum_{l=-5}^5 \chi_{k+l}(\xi_j)}{\sum_{l=-5}^5 \chi_{k+l}(\xi_1) + \sum_{l=-5}^5 \chi_{k+l}(\xi_2)} \cdot \sum_{l=-1}^1 \chi_{k+l}(|\xi|), \quad j = 1, 2.$$

Define the operator  $\Theta_k^j$  on  $L^2(\mathbb{R}^2)$  by  $\widehat{\Theta_k^j f}(\xi) = \beta_k^j(\xi) \hat{f}(\xi)$ ,  $j = 1, 2$ . Let  $\mathbf{e}_1 = (1, 0)$ ,  $\mathbf{e}_2 = (0, 1)$ . Then we have

$$P_k = P_{k,\mathbf{e}_1} \Theta_k^1 + P_{k,\mathbf{e}_2} \Theta_k^2. \quad (2.1)$$

For any  $k \in \mathbb{Z}$ , we define the modulation projectors  $Q_k, Q_{\leq k}, Q_{\geq k}$  on  $L^2(\mathbb{R}^2 \times \mathbb{R})$  by

$$\widehat{Q_k u}(\xi, \tau) = \chi_k(\tau + |\xi|^2) \widehat{u}(\xi, \tau), \quad \widehat{Q_{\leq k} u}(\xi, \tau) = \chi_{\leq k}(\tau + |\xi|^2) \widehat{u}(\xi, \tau),$$

and  $Q_{\geq k} = I - Q_{\leq k-1}$ .

For any  $\mathbf{e} \in \mathbb{S}^1$ , we can decompose  $\mathbb{R}^2 = \lambda \mathbf{e} \oplus H_{\mathbf{e}}$ , where  $H_{\mathbf{e}}$  is the line with normal vector  $\mathbf{e}$ , endowed with the induced measure. For  $1 \leq p, q < \infty$ , we define  $L_{\mathbf{e}}^{p,q}$  the anisotropic Lebesgue space by

$$\|f\|_{L_{\mathbf{e}}^{p,q}} = \left( \int_{\mathbb{R}} \left( \int_{H_{\mathbf{e}} \times \mathbb{R}} |f(\lambda \mathbf{e} + y, t)|^q dy dt \right)^{p/q} d\lambda \right)^{1/p}$$

with the usual definition if  $p = \infty$  or  $q = \infty$ . We write  $L_{\mathbf{e}_1}^{p,q} = L_{x_1}^p L_{x_2,t}^q$ ,  $L_{\mathbf{e}_2}^{p,q} = L_{x_2}^p L_{x_1,t}^q$ .

We use  $\theta \in \mathbb{S}^1$  to denote the spherical variable. Let  $\Delta_\theta$  be the Laplace operator on  $\mathbb{S}^1$ ,  $\partial_\theta$  be the spherical derivative and  $\Lambda_\theta = \sqrt{1 - \Delta_\theta}$ . We identify  $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z}) := \mathbb{T}^1$ . Denote by  $H_\theta^{s,p} = \Lambda_\theta^{-s} L^p$  the standard  $L^p$  Sobolev space on  $\mathbb{T}^1$ . We define  $L_{\mathbf{e}}^{p,q} H_\theta^{s,r}$  by the norm

$$\|f\|_{L_{\mathbf{e}}^{p,q} H_\theta^{s,r}} = \left\| \|f(|x| \cos \theta, |x| \sin \theta, t)\|_{H_\theta^{s,r}} \right\|_{L_{\mathbf{e}}^{p,q}}.$$

By the  $SO(2)$  integration, we will also use the following form

$$\|f\|_{L_{\mathbf{e}}^{p,q} H_\theta^{s,r}} = \left\| \left( \int_0^{2\pi} |\Lambda_\theta^s [f(A_\theta \cdot x, t)]|^r d\theta \right)^{1/r} \right\|_{L_{\mathbf{e}}^{p,q}},$$

where  $A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ . For any function  $f$ , we denote the action  $A_\beta f(x) = f(A_\beta \cdot x)$ . It's easy to see that  $\partial_\beta [A_\beta f] = A_\beta (\partial_\theta f)$ . We use  $\dot{B}_{p,q}^s$  to denote the homogeneous Besov spaces on  $\mathbb{R}^2$  which is the completion of the Schwartz function under the norm

$$\|f\|_{\dot{B}_{p,q}^s} = \left( \sum_{k \in \mathbb{Z}} 2^{qsk} \|P_k f\|_{L^p}^q \right)^{1/q}.$$

We define  $\dot{B}_{p,q,\theta}^{s,\alpha}$  to be the space with the norm  $\|f\|_{\dot{B}_{p,q,\theta}^{s,\alpha}} = \|\Lambda_\theta^\alpha f\|_{\dot{B}_{p,q}^s}$ . Then it's easy to see that  $\|f\|_{\dot{B}_{p,q,\theta}^{s,1}} \sim \|f\|_{\dot{B}_{p,q}^s} + \|\partial_\theta f\|_{\dot{B}_{p,q}^s}$ .

For the Schrödinger map, we need to use the Bourgain-type space associated to the Schrödinger equation. In this paper we use the modulation-homogeneous version as in [3]. We define  $X^{0,b,q}$  to be the completion of the space of Schwartz functions with the norm

$$\|f\|_{X^{0,b,q}} = \left( \sum_{j \in \mathbb{Z}} 2^{jbq} \|Q_j f\|_{L_{t,x}^2}^q \right)^{1/2}. \quad (2.2)$$

If  $q = 2$  we simply write  $X^{0,b} = X^{0,b,2}$ . By the Plancherel's equality we have  $\|f\|_{X^{0,1}} = \|(i\partial_t + \Delta)f\|_{L_{t,x}^2}$ . Since  $X^{0,b,q}$  is not closed under conjugation, we also define the space  $\bar{X}^{0,b,q}$  by the norm  $\|f\|_{\bar{X}^{0,b,q}} = \|\bar{f}\|_{X^{0,b,q}}$ , and similarly write  $\bar{X}^{0,b} = \bar{X}^{0,b,2}$ . It's easy to see that  $X^{0,b,q}$  function is unique modulo solutions of the homogeneous Schrödinger equation. For a more detailed description of the  $X^{0,b,p}$  spaces we refer the readers to [20] and [23].

For any space-time norm  $X$ , we define  $XH_\theta^{s,p}$  by the norm

$$\|f\|_{XH_\theta^{s,p}} = \|A_\theta f\|_{XH_\theta^{s,p}}.$$

We conclude this section by a convolution property of the spherical average space which implies that  $P_k, \Theta_k^1, \Theta_k^2$  are bounded operators in  $XL_\theta^q$  if  $X$  is space translation invariant.

**Lemma 2.1.** *Let  $X$  be a space-time function space on  $\mathbb{R}^2 \times \mathbb{R}$  that is space translation invariant. Then for  $1 \leq q \leq \infty$*

$$\|f * g\|_{XL_\theta^q} \leq C \|f\|_{L_x^1 L_\theta^\infty} \|g\|_{XL_\theta^q}.$$

*Proof.* We have

$$\begin{aligned} \|f * g\|_{XL_\theta^q} &\sim \left\| \int f(A_\beta x - y)g(y)dy \right\|_{XL_\beta^q} \\ &\sim \left\| \int f(A_\beta x - A_\beta y)g(A_\beta y)dy \right\|_{XL_\beta^q} \lesssim \|f\|_{L_x^1 L_\theta^\infty} \|g\|_{XL_\theta^q}, \end{aligned}$$

where in the last inequality we used that  $X$  is space translation invariant.  $\square$

### 3. SPHERICALLY AVERAGED MAXIMAL FUNCTION ESTIMATES

In this section, we prove the spherically averaged maximal function estimate. First, we consider the homogeneous case.

**Lemma 3.1.** *Assume  $k \in \mathbb{Z}$ . Then*

$$\|e^{it\Delta} P_k f\|_{L_{\mathbf{e}}^{2,\infty} L_\theta^2} \lesssim 2^{k/2} \|f\|_2 \quad (3.1)$$

*Proof.* By the scaling invariance, we may assume  $k = 0$ . Moreover, since  $e^{it\Delta}$  commutes with rotation, then by a rotation transform we may assume  $\mathbf{e} = (1, 0)$ . It reduces to prove

$$\|e^{it\Delta} P_0 f\|_{L_{x_1}^2 L_{x_2,t}^\infty L_\theta^2} \lesssim \|f\|_2 \quad (3.2)$$

Using the Hölder inequality and Bernstein's inequality, we easily get

$$\|e^{it\Delta} P_0 f\|_{L_{|x_1| \leq 9}^2 L_{x_2,t}^\infty L_\theta^2} \leq \|e^{it\Delta} P_0 f\|_{L_{x_1,x_2,t}^\infty} \leq C \|f\|_2.$$

Then it remains to show

$$\|e^{it\Delta} P_0 f\|_{L_{|x_1| \geq 9}^2 L_{x_2,t}^\infty L_\theta^2} \leq C \|f\|_2. \quad (3.3)$$

We will prove (3.3) by two steps.

**Step 1.** radial case

We assume  $f$  is radial. It is well known that if  $G(x) = g(|x|)$  is radial and  $G \in L^2(\mathbb{R}^n)$ , then the Fourier transform of  $G$  is also radial (cf. [17]), and

$$\hat{G}(\xi) = 2\pi \int_0^\infty g(s) s^{n-1} (s|\xi|)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(s|\xi|) ds, \quad (3.4)$$

where  $J_m(r)$  is the Bessel function

$$J_m(r) = \frac{(r/2)^m}{\Gamma(m+1/2)\pi^{1/2}} \int_{-1}^1 e^{irt} (1-t^2)^{m-1/2} dt, \quad m > -1/2.$$

Since  $f$  is radial and denote  $\hat{f}(\xi) = h(|\xi|)$ , then by the formula (3.4) we get  $e^{it\Delta} P_0 f(x_1, x_2) = F_0(t, \sqrt{x_1^2 + x_2^2})$ , where for  $\rho \geq 0$

$$F_0(t, \rho) = 2\pi \int_0^\infty e^{-its^2} \eta_0(s) h(s) s J_0(s\rho) ds. \quad (3.5)$$

Therefore, to show (3.3) it is equivalent to show

$$\int_1^\infty \sup_{\rho \geq x, t \in \mathbb{R}} |F_0(t, \rho)|^2 dx \leq C \|h\|_2^2. \quad (3.6)$$

To prove (3.6), we will use the decay properties at the infinity of the Bessel function. More precisely, for  $n \geq 2$

$$J_{\frac{n-2}{2}}(r) = c_n \frac{e^{ir} - e^{-ir}}{r^{1/2}} + c_n r^{\frac{n-2}{2}} e^{-ir} E_+(r) - c_n r^{\frac{n-2}{2}} e^{ir} E_-(r), \quad (3.7)$$

where  $E_{\pm}(r) \lesssim r^{-(n+1)/2}$  if  $r \geq 1$ , see [18]. Inserting (3.7) into (3.5), we then divide  $F_0(t, |x|)$  into two parts: the main term and the error term, namely

$$F_0(t, \rho) = M(t, \rho) + E(t, \rho) \quad (3.8)$$

with

$$\begin{aligned} cM(t, \rho) &= \rho^{-\frac{1}{2}} \int_{\mathbb{R}} \eta_0(s) h(s) s^{\frac{1}{2}} e^{i(\rho s - t s^2)} ds + \rho^{-\frac{1}{2}} \int_{\mathbb{R}} \eta_0(s) h(s) s^{\frac{1}{2}} e^{-i(\rho s + t s^2)} ds, \\ cE(t, \rho) &= \int_{\mathbb{R}} \eta_0(s) h(s) s e^{-i t s^2 - i \rho s} E_+(\rho s) ds - \int_{\mathbb{R}} \eta_0(s) h(s) s e^{-i t s^2 + i \rho s} E_-(\rho s) ds. \end{aligned}$$

For the error term, since  $|E(t, \rho)| \lesssim \rho^{-3/2} \|h\|_2$ , then one get that

$$\int_1^\infty \sup_{\rho \geq x, t \in \mathbb{R}} |E(t, \rho)|^2 dx \lesssim \int_1^\infty x^{-3} \|h\|_2^2 dx \lesssim \|h\|_2^2.$$

It remains to bound the main term. From symmetry, it suffices to show that

$$\int_1^\infty \sup_{\rho \geq x, t \in \mathbb{R}} \frac{1}{\rho} \left| \int_0^\infty \eta_0(s) h(s) e^{i(\rho s - t s^2)} ds \right|^2 dx \lesssim \|h\|_2^2. \quad (3.9)$$

Obviously,

$$\begin{aligned} & \int_1^\infty \sup_{\rho \geq x, t \in \mathbb{R}} \frac{1}{\rho} \left| \int_0^\infty \eta_0(s) h(s) e^{i(\rho s - t s^2)} ds \right|^2 dx \\ & \lesssim \sum_{k=0}^\infty 2^{-k} \int_1^\infty \sup_{2^k x \leq \rho \leq 2^{k+1} x, t \in \mathbb{R}} \frac{1}{x} \left| \int_0^\infty \eta_0(s) h(s) e^{i(\rho s - t s^2)} ds \right|^2 dx. \end{aligned}$$

Define the operator  $T$  acting on  $h \in L^2([1, 3])$  as follows

$$T(h)(x) = \frac{1}{x^{1/2}} \int_0^\infty \eta_0(s) h(s) e^{i(x\rho s - t s^2)} ds.$$

Thus it suffices to show

$$\|Th\|_{L_{x \in [1, \infty]}^2 L_{\rho \sim 2^k, t \in \mathbb{R}}^\infty} \leq C \|h\|_2, \quad \forall k \in \mathbb{Z}_+.$$

By  $TT^*$  argument, it suffices to show

$$\|TT^*f\|_{L_{x \in [1, \infty]}^2 L_{\rho \sim 2^k, t \in \mathbb{R}}^\infty} \leq C \|f\|_{L_{x \in [1, \infty]}^2 L_{\rho \sim 2^k, t \in \mathbb{R}}^1}. \quad (3.10)$$

Indeed, we have

$$TT^*f = \frac{1}{x^{1/2}} \int \left( \int_0^\infty \eta_0^2(s) e^{i((x\rho - x'\rho')s - (t - t')s^2)} ds \right) \frac{1}{x'^{1/2}} f(x', \rho', t') dx' d\rho' dt'.$$

By the stationary phase method, we have

$$\left| \int_0^\infty \eta_0^2(s) e^{i((x\rho - x'\rho')s - (t - t')s^2)} ds \right| \lesssim (1 + |x\rho - x'\rho'|)^{-1/2}.$$

Thus, we get

$$\begin{aligned}
|TT^*f| &\lesssim \frac{1}{x^{1/2}} \int (1 + |x\rho - x'\rho'|)^{-1/2} \frac{1}{x'^{1/2}} |f(x', \rho', t')| dx' d\rho' dt' \\
&\lesssim \int_{|x| \sim |x'|} \frac{|f(x', \rho', t')|}{x^{1/2} x'^{1/2}} dx' d\rho' dt' + \int_{|x| \gg |x'|} \frac{|f(x', \rho', t')|}{2^{k/2} x x'^{1/2}} dx' d\rho' dt' \\
&\quad + \int_{|x| \ll |x'|} \frac{|f(x', \rho', t')|}{2^{k/2} x' x^{1/2}} dx' d\rho' dt' \\
&:= I + II + III.
\end{aligned}$$

Now we show (3.14). For the contribution of the term  $I$ , we have

$$I \lesssim M(\|f(\cdot, \rho, t)\|_{L^1_{\rho, t}})(x)$$

where  $M$  is the Hardy-Littlewood maximal operator. Then from the  $L^2$  boundedness of  $M$ , we see the estimate of  $I$  is fine. The estimate of  $II, III$  simply follows from the Hölder inequality.

**Step 2.** general case

We assume  $f$  is nonradial. First, we make some reductions using the spherical harmonics on  $\mathbb{S}^1$ . For any function  $f \in L^2(\mathbb{R}^2)$ , we can write

$$\widehat{f}(re^{i\theta}) = \sum_{n \in \mathbb{Z}} f_n(r) e^{in\theta}.$$

Hence by the property of Fourier transform (see [17])

$$e^{it\Delta} P_0 f(re^{i\theta}) = \sum_{n \in \mathbb{Z}} 2\pi i^{-n} T_n(f_n)(t, r) e^{in\theta},$$

where

$$T_n(f)(t, r) = \int e^{-it\rho^2} J_n(r\rho) \rho \chi_0(\rho) f(\rho) d\rho.$$

Thus (3.3) becomes

$$\|T_n(f_n)(t, |x|)\|_{L^2_{x_1} L^\infty_{x_2, t} l^2_n} \lesssim \|f_n(|x|)\|_{L^2_{x_n} l^2_n}. \quad (3.11)$$

To prove (3.11), it is equivalent to show

$$\|T_n(f)(t, r)\|_{L^2_x L^\infty_{r \geq |x|, t}} \lesssim \|f\|_{L^2}, \quad (3.12)$$

with a bound independent of  $n \geq 0$

To prove (3.12), we need to use the uniform property of  $J_n$  with respect to  $n$ . We have

$$\begin{aligned}
\|T_n(f)(t, r)\|_{L^2_x L^\infty_{r \geq |x|, t}} &\lesssim \|T_n(f)(t, r)\|_{L^2_{|x| \lesssim n} L^\infty_{r \geq |x|, t}} + \|T_n(f)(t, r)\|_{L^2_{|x| \gg n} L^\infty_{r \geq |x|, t}} \\
&:= A + B.
\end{aligned}$$

First, we estimate the term  $A$ . By the Cauchy-Schwartz inequality, we have

$$A \leq n^{1/2} \|T_n(f)(t, r)\|_{L^\infty_{r \geq 0, t}}.$$

Thus it suffices to show  $|T_n(f)(t, r)| \lesssim n^{-1/2}$ . If  $r \gg n$  or  $r \ll n$ , this follows easily from the fact that  $|J_n(r)| \lesssim n^{-1/2}$ . It remains to show

$$\left\| \int e^{-it\rho^2} J_n(r\rho) \chi_0(\rho) f(\rho) d\rho \right\|_{L^\infty_{r \sim n, t}} \lesssim n^{-1/2} \|f\|_2.$$



By  $TT^*$  argument, it suffices to show

$$\left\| \int \left( \int e^{-i(t-t')\rho^2} J_n(r\rho) J_n(r'\rho) \chi_0(\rho) d\rho \right) g(t', r') dt' dr' \right\|_{L_{r \sim n, t}^\infty} \lesssim n^{-1} \|g\|_{L_{r \sim n, t}^1}.$$

By the uniform decay of Bessel function (e.g. see Lemma 2.2 in [11]),

$$|J_n(r)| \lesssim (1 + |r^2 - n^2|)^{-1/4},$$

it suffices to show

$$\sup_{r, r' \sim 1} \left| \int (1 + |r^2 \rho^2 n^2 - n^2|)^{-1/4} (1 + |r'^2 \rho^2 n^2 - n^2|)^{-1/4} \chi_0(\rho) d\rho \right| \lesssim n^{-1},$$

which follows from the Cauchy-Schwartz inequality.

Now we estimate the term  $B$ . Since  $r \geq |x| \gg n$ , we have (given in [1], for the proof see Lemma 2.5 in [11])

$$J_n(r) = \frac{1}{\sqrt{2\pi}} \frac{e^{i\theta(r)} + e^{-i\theta(r)}}{(r^2 - n^2)^{1/4}} + h(n, r) := J_n^1(r) + J_n^2(r) + J_n^3(r),$$

where

$$\theta(r) = (r^2 - n^2)^{1/2} - \nu \arccos \frac{n}{r} - \frac{\pi}{4}$$

and

$$|h(n, r)| \lesssim r^{-1}.$$

Thus, we get

$$B \leq \sum_{j=1}^3 \|T_n^j(f)(t, r)\|_{L_{|x| \gg n}^2 L_{r \geq |x|, t}^\infty} := \sum_{j=1}^3 B_j$$

where

$$T_n^j(f)(t, r) = \int e^{-it\rho^2} J_n^j(r\rho) \rho \chi_0(\rho) f(\rho) d\rho, \quad j = 1, 2, 3.$$

For  $B_3$ , we use the decay of  $h$  and get

$$\sup_{r \geq |x|} |T_n^3(f)(t, r)| \lesssim \sup_{r \geq |x|} r^{-1} \|f\|_2 \lesssim |x|^{-1} \|f\|_2$$

which suffices to give the bound as desired. It remains to control  $B_1$  since the estimate for  $B_2$  follows in the same way.

It suffices to show that

$$\int_{10n}^\infty \sup_{\rho \geq x, t \in \mathbb{R}} \left| \int_0^\infty \frac{\chi_0(s) e^{i(\theta(s\rho) - ts^2)}}{(\rho^2 s^2 - n^2)^{1/4}} h(s) ds \right|^2 dx \lesssim \|h\|_2^2.$$

Since  $\rho \geq |x| \gg n$  and  $s \sim 1$ , then  $|(\rho^2 s^2 - n^2)^{-1/4} - (\rho s)^{-1/2}| \lesssim |x|^{-5/2} n^2$ , and thus we get

$$\begin{aligned} & \int_{10n}^\infty \sup_{\rho \geq x, t \in \mathbb{R}} \left| \int_0^\infty \left( \frac{1}{(\rho^2 s^2 - n^2)^{1/4}} - \frac{1}{\sqrt{\rho s}} \right) \chi_0(s) e^{i(\theta(s\rho) - ts^2)} h(s) ds \right|^2 dx \\ & \lesssim \int_{10n}^\infty x^{-5} n^4 dx \cdot \|h\|_2^2 \lesssim \|h\|_2^2. \end{aligned}$$

Therefore, it remains to show

$$\int_{10n}^{\infty} \sup_{\rho \geq x, t \in \mathbb{R}} \frac{1}{\rho} \left| \int_0^{\infty} \chi_0(s) e^{i(\theta(s\rho) - ts^2)} h(s) ds \right|^2 dx \lesssim \|h\|_2^2. \quad (3.13)$$

We proceed as in Step 1. Obviously,

$$\begin{aligned} & \int_{10n}^{\infty} \sup_{\rho \geq x, t \in \mathbb{R}} \frac{1}{\rho} \left| \int_0^{\infty} \chi_0(s) e^{i(\theta(s\rho) - ts^2)} h(s) ds \right|^2 dx \\ & \lesssim \sum_{k=0}^{\infty} 2^{-k} \int_{10n}^{\infty} \sup_{2^k x \leq \rho \leq 2^{k+1} x, t \in \mathbb{R}} \frac{1}{x} \left| \int_0^{\infty} \chi_0(s) e^{i(\theta(s\rho) - ts^2)} h(s) ds \right|^2 dx. \end{aligned}$$

Define the operator  $L$  acting on  $h \in L^2([1, 3])$  as follows

$$L(h)(x) = \frac{1}{x^{1/2}} \int_0^{\infty} \eta_0(s) h(s) e^{i(\theta(x\rho s) - ts^2)} ds.$$

Thus it suffices to show

$$\|Lh\|_{L_{x \in [1, \infty]}^2 L_{\rho \sim 2^k, t \in \mathbb{R}}^{\infty}} \leq C \|h\|_2, \quad \forall k \in \mathbb{Z}_+.$$

By  $TT^*$  argument, it suffices to show

$$\|LL^*f\|_{L_{x \in [1, \infty]}^2 L_{\rho \sim 2^k, t \in \mathbb{R}}^{\infty}} \leq C \|f\|_{L_{x \in [1, \infty]}^2 L_{\rho \sim 2^k, t \in \mathbb{R}}^1}. \quad (3.14)$$

Indeed, we have

$$LL^*f = \frac{1}{x^{1/2}} \int \left( \int_0^{\infty} \chi_0^2(s) e^{i(\theta(x\rho s) - \theta(x'\rho's) - (t-t')s^2)} ds \right) \frac{1}{x'^{1/2}} f(x', \rho', t') dx' d\rho' dt'.$$

Direct computation shows that for  $r \gg n$

$$\begin{aligned} \theta'(r) &= (r^2 - n^2)^{1/2} r^{-1} \sim 1, \\ \theta''(r) &= (r^2 - n^2)^{-1/2} - (r^2 - n^2)^{1/2} r^{-2} = (r^2 - n^2)^{-1/2} n^2 r^{-2} \lesssim r^{-1}. \end{aligned}$$

Thus by the stationary phase method, we have

$$\left| \int_0^{\infty} \chi_0^2(s) e^{i(\theta(x\rho s) - \theta(x'\rho's) - (t-t')s^2)} ds \right| \lesssim \begin{cases} 1, & |x| \sim |x'| \\ 2^{-k/2} \max(|x|, |x'|)^{-1/2}, & |x| \neq |x'|. \end{cases}$$

With this the rest proof is the same as in step 1. We complete the proof.  $\square$

Next, we derive the inhomogeneous estimate. Here we use an direct argument of [26] which is in the spirit of Lemma 7.5 in [4].

**Lemma 3.2.** *Let  $k \in \mathbb{Z}$ . Assume  $u, F$  solves the equation*

$$iu_t + \Delta u = F(x, t), \quad u(x, 0) = 0.$$

*Then for any  $\mathbf{e} \in \mathbb{S}^1$  we have*

$$\|P_k u\|_{L_{\mathbf{e}}^{2, \infty} L_{\theta}^2} \lesssim \sup_{\mathbf{e} \in \mathbb{S}^1} \|F\|_{L_{\mathbf{e}}^{1, 2}}. \quad (3.15)$$

*Proof.* By the scaling and rotational invariance, we may assume  $k = 0$  and  $\mathbf{e} = (1, 0)$ .  $P_0 u = U + V$  such that  $\mathcal{F}_x U$  is supported in  $\{|\xi| \sim 1 : |\xi_1| \sim 1\} \times \mathbb{R}$  and  $\mathcal{F}_x V$  is supported in  $\{|\xi| \sim 1 : |\xi_2| \sim 1\} \times \mathbb{R}$ . Thus it suffices to show

$$\|U\|_{L_{x_1}^2 L_{x_2, t}^{\infty} L_{\theta}^2} \lesssim \|F\|_{L_{x_1}^1 L_{x_2, t}^2}, \quad \|V\|_{L_{x_1}^2 L_{x_2, t}^{\infty} L_{\theta}^2} \lesssim \|F\|_{L_{x_2}^1 L_{x_1, t}^2}. \quad (3.16)$$

We only show the estimate for  $U$ , since the estimate for  $V$  is identical. We still write  $u = U$ . We assume  $\mathcal{F}_x F$  is supported in  $\{|\xi| \sim 1 : |\xi_1| \sim 1\} \times \mathbb{R}$ . We have

$$\begin{aligned} u(t, x) &= \int_{\mathbb{R}^3} \frac{e^{it\tau} e^{ix\xi}}{\tau - |\xi|^2} \widehat{F}(\xi, \tau) d\xi d\tau \\ &= \int_{\mathbb{R}^3} \frac{e^{it\tau} e^{ix\xi}}{\tau - |\xi|^2} \widehat{F}(\xi, \tau) (1_{\mathbb{R}^2 \setminus \{(\tau, \xi_2) : \tau - \xi_2^2 \sim 1\}} + 1_{\tau - \xi_2^2 \sim 1}) d\xi d\tau \\ &:= u_1 + u_2. \end{aligned}$$

For  $u_1$ , we simply use the Plancherel equality and get

$$\|\Delta u_1\|_{L^2} + \|\partial_t u_1\|_2 \leq \|F\|_2,$$

and thus by Sobolev embedding and Bernstein's inequality we obtain the desired estimate. Now we estimate  $u_2$ . Let  $G(x_1, \xi_2, \tau) = 1_{|\tau - \xi_2^2| \sim 1} \mathcal{F}_{x_2, t} F$ . Then

$$\begin{aligned} u_2 &= \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{e^{it\tau} e^{ix\xi}}{\tau - |\xi|^2} [e^{-iy_1 \xi_1} G(y_1, \xi_2, \tau)] d\xi d\tau dy_1 \\ &= \int_{\mathbb{R}} T_{y_1}(G(y_1, \cdot))(t, x) dy_1 \end{aligned}$$

where

$$T_{y_1}(f)(t, x) = \int_{\mathbb{R}^3} \frac{e^{it\tau} e^{ix\xi}}{\tau - |\xi|^2} [e^{-iy_1 \xi_1} 1_{|\xi_2| \lesssim 1, \tau - \xi_2^2 \sim 1} f(\xi_2, \tau)] d\xi d\tau.$$

Thus it suffices to prove

$$\|T_{y_1}(f)\|_{L_{x_1}^2 L_{x_2, t}^\infty L_\theta^2} \lesssim \|f\|_{L^2}, \quad \forall y_1 \in \mathbb{R}. \quad (3.17)$$

Define  $s = s(\tau, \xi_2) = \tau - \xi_2^2$ , we have

$$\begin{aligned} T_{y_1}(f)(t, x) &= \int_{\mathbb{R}^2} 1_{|\xi_2| \lesssim 1, \tau - \xi_2^2 \sim 1} \left( \int \frac{e^{i(x_1 - y_1)\xi_1}}{\tau - \xi_2^2 - \xi_1^2} d\xi_1 \right) e^{it\tau} e^{ix_2 \xi_2} f(\xi_2, \tau) d\xi_2 d\tau \\ &= \int_{\mathbb{R}^2} \left( \int \frac{e^{i(x_1 - y_1)\xi_1}}{2\sqrt{s}} \left( \frac{1}{\sqrt{s} + \xi_1} + \frac{1}{\sqrt{s} - \xi_1} \right) d\xi_1 \right) \\ &\quad \cdot e^{it\tau} e^{ix_2 \xi_2} 1_{|\xi_2| \lesssim 1, \tau - \xi_2^2 \sim 1} f(\xi_2, \tau) d\xi_2 d\tau \\ &:= I_1(f) + I_2(f). \end{aligned}$$

We only estimate  $I_1$ , since  $I_2$  follows in the same way. By the property of Hilbert transform, we get

$$I_1(f)(t, x) = \int_{\mathbb{R}^2} \frac{e^{-i(x_1 - y_1)\sqrt{s}}}{2\sqrt{s}} i \operatorname{sgn}(x_1 - y_1) \cdot e^{it\tau} e^{ix_2 \xi_2} 1_{|\xi_2| \lesssim 1, \tau - \xi_2^2 \sim 1} f(\xi_2, \tau) d\xi_2 d\tau.$$

Making a change of variable  $\eta_1 = -\sqrt{\tau - \xi_2^2}$ ,  $d\tau = 2\eta_1 d\eta_1$ , we obtain

$$\begin{aligned} I_1(f)(t, x) &= i \operatorname{sgn}(x_1 - y_1) \int_{\mathbb{R}^2} e^{it(\eta_1^2 + \xi_2^2)} e^{i(x_1 \eta_1 + x_2 \xi_2)} \\ &\quad \cdot e^{-iy_1 \eta_1} 1_{|\xi_2| \lesssim 1, \eta_1 \sim 1} f(\xi_2, \eta_1^2 + \xi_2^2) d\xi_2 d\eta_1. \end{aligned}$$

Thus, by Lemma 3.1 we get

$$\|I_1(f)\|_{L_{x_1}^2 L_{x_2, t}^\infty L_\theta^2} \lesssim \|1_{|\xi_2| \lesssim 1} \cdot 1_{\eta_1 \sim 1} f(\xi_2, \eta_1^2 + \xi_2^2)\|_{L^2} \lesssim \|f\|_2.$$

We complete the proof of the lemma.  $\square$

## 4. CUBIC DERIVATIVE NLS

In this section we prove Theorem 1.2. The ideas is from [4]. First, we define the main dyadic function space  $F_k$  and  $N_k$  for  $k \in \mathbb{Z}$ . If  $f(x, t) \in L^2(\mathbb{R}^2 \times \mathbb{R})$  has spatial frequency localized in  $\{|\xi| \sim 2^k\}$ , define

$$\begin{aligned} \|f\|_{F_k} &= \|f\|_{L_t^\infty L_x^2} + \|f\|_{L_t^4 L_x^4} + 2^{k/6} \sup_{|j-k| \leq 20} \sup_{\mathbf{e} \in \mathbb{S}^1} \|P_{j,\mathbf{e}} f\|_{L_{\mathbf{e}}^{6,3}} \\ &\quad + 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^1} \|f\|_{L_{\mathbf{e}}^{2,\infty} L_\theta^2} + 2^{k/2} \sup_{|j-k| \leq 20} \sup_{\mathbf{e} \in \mathbb{S}^1} \|P_{j,\mathbf{e}}(A_\beta f)\|_{L_{\mathbf{e}}^{\infty,2} L_\beta^2}, \\ \|f\|_{G_k} &= \|f\|_{L_t^\infty L_x^2} + \|f\|_{L_t^4 L_x^4} + 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^1} \|f\|_{L_{\mathbf{e}}^{2,\infty} L_\theta^2}, \\ \|f\|_{N_k} &= \inf_{f=f_1+f_2+f_3+f_4} (\|f_1\|_{L_{t,x}^{4/3}} + 2^{k/6} \|f_2\|_{L_{\mathbf{e}_1}^{3/2,6/5}} \\ &\quad + 2^{k/6} \|f_3\|_{L_{\mathbf{e}_2}^{3/2,6/5}} + 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^1} \|f_4\|_{L_{\mathbf{e}}^{1,2}}). \end{aligned}$$

Then we define the space  $F^s, N^s$  with the following norm

$$\begin{aligned} \|u\|_{F^s} &= \sum_{k \in \mathbb{Z}} 2^{ks} (\|P_k u\|_{F_k} + \|P_k \partial_\theta u\|_{F_k}) := \sum_{k \in \mathbb{Z}} 2^{ks} \|P_k u\|_{\tilde{F}_k}, \\ \|u\|_{G^s} &= \sum_{k \in \mathbb{Z}} 2^{ks} (\|P_k u\|_{G_k} + \|P_k \partial_\theta u\|_{G_k}) := \sum_{k \in \mathbb{Z}} 2^{ks} \|P_k u\|_{\tilde{G}_k}, \\ \|u\|_{N^s} &= \sum_{k \in \mathbb{Z}} 2^{ks} (\|P_k u\|_{N_k} + \|P_k \partial_\theta u\|_{N_k}) := \sum_{k \in \mathbb{Z}} 2^{ks} \|P_k u\|_{\tilde{N}_k}. \end{aligned}$$

Note that to use the spherically averaged maximal function estimate, we need the spherically averaged local smoothing estimate.

**Lemma 4.1** (Linear estimates). *Assume  $u, F, u_0$  solves the following equation*

$$i\partial_t u + \Delta u = F, \quad u(0, x) = u_0.$$

*Then for any  $s \in \mathbb{R}$ , we have*

$$\|u\|_{F^s} = \|u_0\|_{\dot{B}_{2,1,\theta}^{s,1}} + \|F\|_{N^s}.$$

*Proof.* By the definition, it suffices to show

$$\|P_k u\|_{F_k} \lesssim \|P_k u_0\|_2 + \|P_k F\|_{N_k}. \quad (4.1)$$

Since  $\Delta$  commutes with rotation and the local smoothing estimate (see [12]), we have

$$\begin{aligned} \|P_{k,\mathbf{e}} A_\theta u\|_{L_{\mathbf{e}}^{\infty,2} L_\theta^2} &\lesssim \|P_{k,\mathbf{e}} A_\theta u\|_{L_\theta^2 L_{\mathbf{e}}^{\infty,2}} \lesssim 2^{-k/2} \|A_\theta u_0\|_{L_\theta^2 L^2} + 2^{-k} \sup_{\mathbf{e} \in \mathbb{S}^1} \|A_\theta F\|_{L_{\mathbf{e}}^{1,2}} \|L_\theta^2 \\ &\lesssim 2^{-k/2} \|u_0\|_2 + 2^{-k} \sup_{\mathbf{e} \in \mathbb{S}^1} \|F\|_{L_{\mathbf{e}}^{1,2}}. \end{aligned}$$

Similarly, in the above inequality we can replace  $\sup_{\mathbf{e} \in \mathbb{S}^1} \|F\|_{L_{\mathbf{e}}^{1,2}}$  by  $\sup_{\mathbf{e} \in \mathbb{S}^1} \|F\|_{L_{\mathbf{e}}^{3/2,6/5}}$  and  $\|F\|_{L_{x,t}^{4/3}}$ . The other components except for the maximal function follow from the known linear estimate. For the maximal function component, we use Lemma 3.1-3.2 and the Christ-Kiselev lemma [7] (or Lemma 7.3 in [4]).  $\square$

To prove Theorem 1.2, by the standard iteration method, it suffices to show the trilinear estimates. We need the following lemma.

**Lemma 4.2.** *Assume  $k_1, k_2 \in \mathbb{Z}$ . Then*

$$\begin{aligned} & \|P_{k_1} u P_{k_2} \bar{v}\|_{L_{t,x}^2} + \|P_{k_1} \partial_\theta u P_{k_2} \bar{v}\|_{L_{t,x}^2} + \|P_{k_1} u P_{k_2} \partial_\theta \bar{v}\|_{L_{t,x}^2} \\ & \lesssim 2^{k_1/2} 2^{-k_2/2} \|P_{k_1} u\|_{\tilde{G}_{k_1}} \|P_{k_2} v\|_{\tilde{F}_{k_2}}. \end{aligned}$$

*Proof.* Since  $A_\beta$  commute with  $P_k$ , and by Lemma 2.1 we have

$$\begin{aligned} & \|P_{k_1} u P_{k_2} \bar{v}\|_{L_{t,x}^2} \\ & = \|P_{k_1} A_\beta u P_{k_2} A_\beta \bar{v}\|_{L_{t,x}^2 L_\beta^2} \\ & \lesssim \|P_{k_1} A_\beta u P_{k_2, \mathbf{e}_1} \Theta_{k_2}^1 A_\beta \bar{v}\|_{L_{t,x}^2 L_\beta^2} + \|P_{k_1} A_\beta u P_{k_2, \mathbf{e}_2} \Theta_{k_2}^2 A_\beta \bar{v}\|_{L_{t,x}^2 L_\beta^2} \\ & \lesssim \|P_{k_1} A_\beta u\|_{L_{\mathbf{e}_1}^{2,\infty} L_\beta^\infty} \|P_{k_2, \mathbf{e}_1} A_\beta \bar{v}\|_{L_{\mathbf{e}_1}^{\infty,2} L_\beta^2} + \|P_{k_1} A_\beta u\|_{L_{\mathbf{e}_2}^{2,\infty} L_\beta^\infty} \|P_{k_2, \mathbf{e}_2} A_\beta \bar{v}\|_{L_{\mathbf{e}_2}^{\infty,2} L_\beta^2} \\ & \lesssim 2^{k_1/2} 2^{-k_2/2} \|P_{k_1} u\|_{\tilde{G}_{k_1}} \|P_{k_2} v\|_{\tilde{F}_{k_2}}. \end{aligned}$$

For the other component, we have

$$\begin{aligned} & \|P_{k_1} \partial_\theta u P_{k_2} \bar{v}\|_{L_{t,x}^2} + \|P_{k_1} u P_{k_2} \partial_\theta \bar{v}\|_{L_{t,x}^2} \\ & = \|P_{k_1} \partial_\beta (A_\beta u) \cdot P_{k_2} A_\beta \bar{v}\|_{L_{t,x}^2 L_\beta^2} + \|P_{k_1} (A_\beta u) \cdot P_{k_2} \partial_\beta (A_\beta \bar{v})\|_{L_{t,x}^2 L_\beta^2} \\ & := I + II. \end{aligned}$$

For  $I$ , by the Sobolev embedding  $H_\theta^1(\mathbb{T}^1) \hookrightarrow L_\theta^\infty(\mathbb{T}^1)$  we have

$$\begin{aligned} I & \lesssim \|P_{k_1} \partial_\beta (A_\beta u) \cdot P_{k_2, \mathbf{e}_1} A_\beta \bar{v}\|_{L_{t,x}^2 L_\beta^2} + \|P_{k_1} \partial_\beta (A_\beta u) \cdot P_{k_2, \mathbf{e}_2} A_\beta \bar{v}\|_{L_{t,x}^2 L_\beta^2} \\ & \lesssim \|P_{k_1} \partial_\beta (A_\beta u)\|_{L_{\mathbf{e}_1}^{2,\infty} L_\beta^2} \|P_{k_2, \mathbf{e}_1} A_\beta \bar{v}\|_{L_{\mathbf{e}_1}^{\infty,2} L_\beta^\infty} \\ & \quad + \|P_{k_1} \partial_\beta (A_\beta u)\|_{L_{\mathbf{e}_2}^{2,\infty} L_\beta^2} \|P_{k_2, \mathbf{e}_2} A_\beta \bar{v}\|_{L_{\mathbf{e}_2}^{\infty,2} L_\beta^\infty} \\ & \lesssim 2^{k_1/2} 2^{-k_2/2} \|P_{k_1} u\|_{\tilde{G}_{k_1}} \|P_{k_2} v\|_{\tilde{F}_{k_2}}. \end{aligned}$$

For  $II$ , we have

$$\begin{aligned} II & \lesssim \|P_{k_1} (A_\beta u) \cdot P_{k_2, \mathbf{e}_1} \partial_\beta (A_\beta \bar{v})\|_{L_{t,x}^2 L_\beta^2} + \|P_{k_1} (A_\beta u) \cdot P_{k_2, \mathbf{e}_2} \partial_\beta (A_\beta \bar{v})\|_{L_{t,x}^2 L_\beta^2} \\ & \lesssim \|P_{k_1} (A_\beta u)\|_{L_{\mathbf{e}_1}^{2,\infty} L_\beta^\infty} \|P_{k_2, \mathbf{e}_1} \partial_\beta (A_\beta \bar{v})\|_{L_{\mathbf{e}_1}^{\infty,2} L_\beta^2} \\ & \quad + \|P_{k_1} (A_\beta u)\|_{L_{\mathbf{e}_2}^{2,\infty} L_\beta^\infty} \|P_{k_2, \mathbf{e}_2} \partial_\beta (A_\beta \bar{v})\|_{L_{\mathbf{e}_2}^{\infty,2} L_\beta^2} \\ & \lesssim 2^{k_1/2} 2^{-k_2/2} \|P_{k_1} u\|_{\tilde{G}_{k_1}} \|P_{k_2} v\|_{\tilde{F}_{k_2}}. \end{aligned}$$

We complete the proof of the lemma.  $\square$

**Lemma 4.3** (Nonlinear estimates). *Assume  $i = 1, 2$ ,  $s \geq 1/2$ . Then*

$$\|u \bar{v} \partial_{x_i} w\|_{N^s} \lesssim \|u\|_{F^s} \|v\|_{F^{1/2}} \|w\|_{F^{1/2}} + \|u\|_{F^{1/2}} \|v\|_{F^s} \|w\|_{F^{1/2}} + \|u\|_{F^{1/2}} \|v\|_{F^{1/2}} \|w\|_{F^s}.$$

*Proof.* We only prove the case  $s = 1/2$ , since the other case are similar. By the definition, we have

$$\begin{aligned} & \|u \bar{v} \partial_{x_i} w\|_{N^{1/2}} \\ & = \sum_{k_4} 2^{k_4/2} (\|P_{k_4} [u \bar{v} \partial_{x_i} w]\|_{N_{k_4}} + \|\partial_\theta P_{k_4} [u \bar{v} \partial_{x_i} w]\|_{N_{k_4}}) \\ & \leq \sum_{k_1, k_2, k_3, k_4} 2^{k_4/2} (\|P_{k_4} [P_{k_1} u P_{k_2} \bar{v} \partial_{x_i} P_{k_3} w]\|_{N_{k_4}} + \|\partial_\theta P_{k_4} [P_{k_1} u P_{k_2} \bar{v} \partial_{x_i} P_{k_3} w]\|_{N_{k_4}}) \\ & := I + II. \end{aligned}$$

We will estimate the sum above case by case, according to the type of frequency interactions. By symmetry, we may assume  $k_1 \leq k_2$ . We also assume that  $k_2 \leq k_3$ , namely the derivative falls on the largest frequency, since the other case  $k_2 > k_3$  can be handled similarly.

**Case 1:**  $k_4 \leq k_1 + 200$ .

For this case, we use the Strichartz norm  $L^{4/3}$  for  $N_{k,\alpha}$ . By the properties of Fourier support of input functions, we may assume  $k_3 \leq k_2 + 300$ . Thus we have

$$\begin{aligned} I &\lesssim \sum_{k_i: k_4 \leq \min(k_1, k_2, k_3) + 5} 2^{k_4/2} \|P_{k_4}[P_{k_1}uP_{k_2}\bar{v}\partial_{x_i}P_{k_3}w]\|_{L_{t,x}^{4/3}} \\ &\lesssim \sum_{k_i: k_4 \leq k_1 + 5} 2^{k_4/2} \|P_{k_1}u\|_{L_{T,x}^4} 2^{k_2/2} \|P_{k_2}v\|_{L_{T,x}^4} 2^{k_3/2} \|P_{k_3}w\|_{L_{t,x}^4} \\ &\lesssim \|u\|_{F^{1/2}} \|v\|_{F^{1/2}} \|w\|_{F^{1/2}}. \end{aligned}$$

The estimate for  $II$  is the same as  $I$ , since  $\partial_\theta$  commutes with  $P_k$ .

**Case 2:**  $k_1 + 200 < k_4 \leq k_2 + 100$ .

In this case we have  $k_1 < k_2 - 100$  and  $k_3 \leq k_2 + 200$ . Then we get

$$\begin{aligned} I &\lesssim \sum_{k_i} 2^{k_4/2} 2^{k_4/6} (\|P_{k_4}[P_{k_1}uP_{k_2}\bar{v}\partial_{x_i}P_{k_3,\mathbf{e}_1}w]\|_{L_{\mathbf{e}_1}^{3/2,6/5}} \\ &\quad + \|P_{k_4}[P_{k_1}uP_{k_2}\bar{v}\partial_{x_i}P_{k_3,\mathbf{e}_2}w]\|_{L_{\mathbf{e}_2}^{3/2,6/5}}) := I_1 + I_2. \end{aligned}$$

By symmetry we only estimate  $I_1$ . By Lemma 4.2 we get

$$\begin{aligned} I_1 &\lesssim \sum_{k_i} 2^{k_4/2} 2^{k_4/6} 2^{k_3} \|P_{k_1}uP_{k_2}\bar{v}\|_{L_{t,x}^2} \|P_{k_3,\mathbf{e}_1}w\|_{L_{\mathbf{e}_1}^{6,3}} \\ &\lesssim \sum_{k_i} 2^{k_1/2} 2^{k_2/2} 2^{k_3/2} \|P_{k_1}u\|_{\tilde{F}_{k_1}} \|P_{k_2}v\|_{\tilde{F}_{k_2}} \|P_{k_3}w\|_{\tilde{F}_{k_2}} \\ &\lesssim \|u\|_{F^{1/2}} \|v\|_{F^{1/2}} \|w\|_{F^{1/2}}. \end{aligned}$$

For the term  $II$ , we have

$$\begin{aligned} II &\lesssim \sum_{k_i} \|P_{k_4}[\partial_\theta(P_{k_1}uP_{k_2}\bar{v})\partial_{x_i}P_{k_3}w]\|_{N_{k_4}} + \sum_{k_i} \|P_{k_4}[P_{k_1}uP_{k_2}\bar{v}\partial_\theta\partial_{x_i}P_{k_3}w]\|_{N_{k_4}} \\ &:= II_1 + II_2. \end{aligned}$$

For the term  $II_1$ , as for the term  $I$  we get

$$\begin{aligned} II_1 &\lesssim \sum_{k_i} 2^{k_4/2} 2^{k_3} \|\partial_\theta(P_{k_1}uP_{k_2}\bar{v})\|_{L_{t,x}^2} \|P_{k_3}w\|_{\tilde{F}_{k_3}} \\ &\lesssim \|u\|_{F^{1/2}} \|v\|_{F^{1/2}} \|w\|_{F^{1/2}}. \end{aligned}$$

It remains to estimate the term  $II_2$ . Note that  $[\partial_\theta, \partial_{x_i}] = \partial_{x_i}$ , so we get

$$II_2 \lesssim \sum_{k_i} \|P_{k_4}[P_{k_1}uP_{k_2}\bar{v}\partial_{x_i}P_{k_3}w]\|_{N_{k_4}} + \sum_{k_i} \|P_{k_4}[P_{k_1}uP_{k_2}\bar{v}\partial_{x_i}P_{k_3}\partial_\theta w]\|_{N_{k_4}}.$$

The first term on the righthand side above is just  $I$ , while the second term can be handled exactly as for  $I$ .

**Case 3:**  $k_4 > \max(k_1 + 200, k_2 + 100)$ .

In this case we have  $|k_4 - k_3| \leq 5$ . For the term  $I$  by Lemma 4.2 and noting that  $\|f\|_{L_e^{2,\infty}} \lesssim \|f\|_{L_e^{2,\infty} L_\theta^\infty}$ , we get

$$\begin{aligned} I &\lesssim \sum_{k_i} 2^{k_4/2} 2^{-k_4/2} \|P_{k_4}[P_{k_1} u P_{k_2} \bar{v} \partial_{x_i} P_{k_3} w]\|_{L_e^{1,2}} \\ &\lesssim \sum_{k_i} \|P_{k_1} u P_{k_3} \partial_{x_i} w\|_{L_{x,t}^2} \|P_{k_2} v\|_{L_e^{2,\infty}} \\ &\lesssim \|u\|_{F^{1/2}} \|v\|_{F^{1/2}} \|w\|_{F^{1/2}}. \end{aligned}$$

For the term  $II$ , we have

$$\begin{aligned} II &\lesssim \sum_{k_i} \|P_{k_4}[P_{k_1}(\partial_\theta u) P_{k_2} \bar{v} \partial_{x_i} P_{k_3} w]\|_{L_e^{1,2}} + \sum_{k_i} \|P_{k_4}[P_{k_1} u (P_{k_2} \partial_\theta \bar{v}) \partial_{x_i} P_{k_3} w]\|_{L_e^{1,2}} \\ &\quad + \sum_{k_i} \|P_{k_4}[P_{k_1} u P_{k_2} \bar{v} \partial_\theta \partial_{x_i} P_{k_3} w]\|_{L_e^{1,2}} := II_1 + II_2 + II_3. \end{aligned}$$

For the term  $II_1$  we have

$$II_1 \lesssim \sum_{k_i} \|P_{k_1}(\partial_\theta u) P_{k_3} \partial_{x_i} w\|_{L_{x,t}^2} \|P_{k_2} v\|_{L_e^{2,\infty}} \lesssim \|u\|_{F^{1/2,1}} \|v\|_{F^{1/2,1}} \|w\|_{F^{1/2,1}}.$$

Similarly, we can bound the term  $II_2$ . For the term  $II_3$ , we use the commutator as in Case 2 and then bound as  $II_1$ . Thus we finish the proof.  $\square$

## 5. SCHRÖDINGER MAP IN TWO DIMENSIONS

In this section, we prove Theorem 1.3. Consider the Schrödinger maps

$$\partial_t s = s \times \Delta_x s, \quad s(0) = s_0, \quad (5.1)$$

where  $s : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ . Using the stereographic projection

$$u = \frac{s_1 + i s_2}{1 + s_3},$$

we see  $u$  solves the equation

$$i \partial_t u + \Delta u = \frac{2\bar{u}}{1 + |u|^2} \sum_{i=1}^n (\partial_{x_i} u)^2. \quad (5.2)$$

Conversely, if  $u$  solves (5.2), then

$$s = \left( \frac{2\Re u}{1 + |u|^2}, \frac{2\Im u}{1 + |u|^2}, \frac{1 - |u|^2}{1 + |u|^2} \right)$$

solves (5.1). Now we focus on the study of (5.2).

We define the main dyadic function space  $Z_k$  and  $W_k$  for  $k \in \mathbb{Z}$ . If  $f(x, t) \in L^2(\mathbb{R}^2 \times \mathbb{R})$  has spatial frequency localized in  $\{|\xi| \sim 2^k\}$ , define

$$\begin{aligned} \|f\|_{Z_k} &= \|f\|_{X^{0,1/2,\infty}} + 2^{-k} \|f\|_{X^{0,1}} + \|f\|_{L_t^\infty L_x^2} + \|f\|_{L_t^4 L_x^4} + 2^{k\varepsilon/2} \|f\|_{L_t^4 L_x^{\frac{4}{1+\varepsilon}} L_\theta^3} \\ &\quad + 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^1} \|f\|_{L_{\mathbf{e}}^{2,\infty} L_\theta^2} + 2^{k/2} \sup_{|j-k| \leq 20} \sup_{\mathbf{e} \in \mathbb{S}^1} \|P_{j,\mathbf{e}}(A_\beta f)\|_{L_{\mathbf{e}}^{\infty,2} L_\beta^2}, \\ \|f\|_{Y_k} &= \|f\|_{L_t^\infty L_x^2} + \|f\|_{L_t^4 L_x^4} + 2^{k\varepsilon/2} \|f\|_{L_t^4 L_x^{\frac{4}{1+\varepsilon}} L_\theta^3} + 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^1} \|f\|_{L_{\mathbf{e}}^{2,\infty} L_\theta^2} \\ &\quad + 2^{-k} \inf_{f=f_1+f_2} (\|f_1\|_{X^{0,1}} + \|f_2\|_{\bar{X}^{0,1}}), \\ \|f\|_{W_k} &= \inf_{f=f_1+f_2+f_3+f_4} (\|f_1\|_{L_{t,x}^{4/3}} + \|f_2\|_{L_t^1 L_x^2} + 2^{-k/2} \sup_{\mathbf{e} \in \mathbb{S}^1} \|f_3\|_{L_{\mathbf{e}}^{1,2}} + \|f_4\|_{X^{0,-1/2,1}}) \\ &\quad + 2^{-k} \|f\|_{L_{t,x}^2}, \end{aligned}$$

where  $0 < \varepsilon \ll 1$  will be a fixed universal number (e.g.  $\varepsilon < 0.01$  would work). Then we define the space  $Z^s, W^s$  with the following norm

$$\begin{aligned} \|u\|_{Z^s} &= \sum_{k \in \mathbb{Z}} 2^{ks} (\|P_k u\|_{Z_k} + \|P_k \partial_\theta u\|_{Z_k}) := \sum_{k \in \mathbb{Z}} 2^{ks} \|P_k u\|_{\tilde{Z}_k}, \\ \|u\|_{Y^s} &= \sum_{k \in \mathbb{Z}} 2^{ks} (\|P_k u\|_{Y_k} + \|P_k \partial_\theta u\|_{Y_k}) := \sum_{k \in \mathbb{Z}} 2^{ks} \|P_k u\|_{\tilde{Y}_k}, \\ \|u\|_{W^s} &= \sum_{k \in \mathbb{Z}} 2^{ks} (\|P_k u\|_{W_k} + \|P_k \partial_\theta u\|_{W_k}) := \sum_{k \in \mathbb{Z}} 2^{ks} \|P_k u\|_{\tilde{W}_k}. \end{aligned}$$

To prove Theorem 1.3, it suffices to prove

**Theorem 5.1.** *Assume  $n = 2$ ,  $u_0 \in \dot{B}_{2,1,\theta}^{1,1}$  with  $\|u_0\|_{\dot{B}_{2,1,\theta}^{1,1}} = \varepsilon_0 \ll 1$ . Then there exists a unique global solution  $u$  to (5.2) such that  $\|u\|_{Z^1} \lesssim \varepsilon_0$ . Moreover, the map  $u_0 \rightarrow u$  is Lipschitz from  $\dot{B}_{2,1,\theta}^{1,1}$  to  $C(\mathbb{R}; \dot{B}_{2,1,\theta}^{1,1})$ , and scattering holds in this space.*

We will prove the above theorem via picard iteration argument. We need to prove some linear estimates and nonlinear estimates.

**Lemma 5.2** (Linear estimates). *Assume  $u, f, u_0$  solves the following equation*

$$(i\partial_t + \Delta)u = f, \quad u(0) = u_0.$$

*Then we have*

$$\|u\|_{Z_k} \lesssim \|u_0\|_{L_x^2} + \|f\|_{W_k}. \quad (5.3)$$

*Proof.* Most of the estimates were given in [4]. We only need to deal with the component  $2^{k\varepsilon/2} \|f\|_{L_t^4 L_x^{\frac{4}{1+\varepsilon}} L_\theta^3}$ . We will use the generalised Strichartz estimates proved in [11]. In [11] the author proved the following estimate

$$\|e^{it\Delta} P_0 u_0\|_{L_t^2 L_x^{6+} L_\theta^2} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)}.$$

Interpolating the above with the following estimate proved in [14] (Theorem 3.1):

$$\|e^{it\Delta} P_0 u_0\|_{L_t^2 L_x^\infty L_\theta^p} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)}, \quad 1 \leq p < \infty,$$

we get

$$\|e^{it\Delta} P_0 u_0\|_{L_t^2 L_x^{\frac{2}{\varepsilon}} L_\theta^6} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)}.$$



Interpolating above with the trivial estimate  $\|e^{it\Delta}P_0u_0\|_{L_t^\infty L_x^2 L_\theta^2} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)}$ , we get

$$\|e^{it\Delta}P_0u_0\|_{L_t^4 L_x^{\frac{4}{1+\varepsilon}} L_\theta^3} \lesssim \|u_0\|_{L^2(\mathbb{R}^2)}.$$

Then by scaling transform we complete the proof.  $\square$

We use Taylor's expansion to rewrite the nonlinear term: if  $\|u\|_\infty < 1$

$$\frac{2\bar{u}}{1+|u|^2} \sum_{i=1}^n (\partial_{x_i} u)^2 = \sum_{k=0}^{\infty} 2\bar{u}(-|u|^2)^k \sum_{i=1}^n (\partial_{x_i} u)^2.$$

We prove

**Lemma 5.3** (Nonlinear estimates). *The following estimates holds*

$$\|\bar{u}(-|u|^2)^k \sum_{i=1}^n (\partial_{x_i} u)^2\|_{W^1} \lesssim C^{2k} \|u\|_{Y^1}^{2k+1} \|u\|_{Z^1} \|u\|_{Z^1}.$$

The above lemma will follow from the following lemmas.

**Lemma 5.4.** (1) *If  $j \geq 2k - 100$  and  $X$  is a space-time translation invariant Banach space, then  $Q_{\leq j}P_k$  is bounded on  $X$  with bound independent of  $j, k$ .*

(2) *For any  $j, k$ ,  $Q_{\leq j}P_{k,\mathbf{e}}$  is bounded on  $L_{\mathbf{e}}^{p,2}$  and  $Q_{\leq j}$  is bounded on  $L_t^p L_x^2$  for  $1 \leq p \leq \infty$ , with bound independent of  $j, k$ .*

*Proof.* (1) The operator  $Q_{\leq j}P_k$  corresponds to the space-time multiplier with symbol  $\eta(\frac{\tau+|\xi|^2}{2^j})\chi(\frac{|\xi|}{2^k})$ . It suffices to show

$$\|\mathcal{F}^{-1}\eta(\frac{\tau+|\xi|^2}{2^j})\chi(\frac{|\xi|}{2^k})\|_{L_{t,x}^1} \lesssim 1, \quad \forall j, k.$$

From integration by part and the condition  $j \geq 2k - 100$ , we get

$$|\int_{\mathbb{R}^n \times \mathbb{R}} \eta(\frac{\tau+|\xi|^2}{2^j})\chi(\frac{|\xi|}{2^k}) e^{ix\xi} e^{it\tau} d\xi d\tau| \lesssim 2^j (1+2^j|t|)^{-2} \cdot 2^k (1+2^k|x|)^{-n-1},$$

which completes the proof.

(2) This was proved in [3]. In view of part (1), we may assume  $j \leq 2k - 100$ . By rotation, we may take  $\mathbf{e} = \mathbf{e}_1$ . By Plancherel's equality, it suffices to prove

$$\left\| \int_{\mathbb{R}} e^{ix_1\xi_1} \eta(\frac{\tau+|\xi|^2}{2^j}) \chi(\frac{|\xi|}{2^k}) \chi(\frac{|\xi_1|}{2^k}) (\mathcal{F}_{x_1} f)(\xi, \tau) d\xi_1 \right\|_{L_{x_1}^p L_{\tau,\bar{\xi}}^2} \lesssim \|f\|_{L_{x_1}^p L_{\tau,\bar{\xi}}^2}.$$

Furthermore, it suffices to show

$$\sup_{\bar{\xi}, \tau} \left\| \int_{\mathbb{R}} e^{ix_1\xi_1} \eta(\frac{\tau+|\xi|^2}{2^j}) \chi(\frac{|\xi|}{2^k}) \chi(\frac{|\xi_1|}{2^k}) d\xi_1 \right\|_{L_{x_1}^1} \lesssim 1. \quad (5.4)$$

For fixed  $\bar{\xi}, \tau$ , since  $|\tau+|\bar{\xi}|^2+\xi_1^2| \lesssim 2^j \ll |\xi_1|^2$ , thus  $\tau+|\bar{\xi}|^2$  is negative, and we have either  $|\xi_1 - \sqrt{-\tau-|\bar{\xi}|^2}| \lesssim 2^{j-k}$  or  $|\xi_1 + \sqrt{-\tau-|\bar{\xi}|^2}| \lesssim 2^{j-k}$ . Thus  $\xi_1$  varies in a ball of size  $2^{j-k}$ . From integration by part and the condition  $j \leq 2k - 100$ , we get

$$|\int_{\mathbb{R}} e^{ix_1\xi_1} \eta(\frac{\tau+|\xi|^2}{2^j}) \chi(\frac{|\xi|}{2^k}) \chi(\frac{|\xi_1|}{2^k}) d\xi_1| \lesssim 2^{j-k} (1+2^{j-k}|x_1|)^{-2}.$$

To see  $Q_{\leq j}$  is bounded on  $L_t^p L_x^2$ , by Plancherel's equality it suffices to prove

$$\left\| \int e^{it\tau} \eta\left(\frac{\tau - \xi^2}{2^j}\right) \mathcal{F}_t f(\tau, \xi) d\tau \right\|_{L_t^p L_\xi^2} \lesssim \|f\|_{L_t^p L_\xi^2},$$

which is equivalent to show

$$\left\| \int e^{it\tau} \eta\left(\frac{\tau + \xi^2}{2^j}\right) (\mathcal{F}_t f)(\tau + |\xi|^2, \xi) d\tau \right\|_{L_t^p L_\xi^2} \lesssim \|f\|_{L_t^p L_\xi^2}.$$

The above inequality is trivial.  $\square$

**Lemma 5.5.** *Assume  $k_1, k_2, k_3 \in \mathbb{Z}$ . Then*

$$\begin{aligned} & \|P_{k_3}(P_{k_1}uP_{k_2}v)\|_{L_{t,x}^2} + \|P_{k_3}(P_{k_1}\partial_\theta uP_{k_2}v)\|_{L_{t,x}^2} + \|P_{k_3}(P_{k_1}uP_{k_2}\partial_\theta v)\|_{L_{t,x}^2} \\ & \lesssim 2^{\varepsilon[\min(k_1, k_2, k_3) - \max(k_1, k_2, k_3)]/2} \|P_{k_1}u\|_{\tilde{Y}_{k_1}} \|P_{k_2}v\|_{\tilde{Y}_{k_2}}. \end{aligned}$$

*Proof.* We only estimate  $\|P_{k_3}(P_{k_1}\partial_\theta uP_{k_2}v)\|_{L_{t,x}^2}$ . If  $k_3 \leq \min(k_1, k_2)$ , then

$$\begin{aligned} \|P_{k_3}(P_{k_1}\partial_\theta uP_{k_2}v)\|_{L_{t,x}^2} & \lesssim 2^{\varepsilon k_3} \|P_{k_1}\partial_\theta uP_{k_2}v\|_{L_t^2 L_x^{\frac{2}{1+\varepsilon}}} \\ & \lesssim 2^{\varepsilon k_3} 2^{-\varepsilon(k_1+k_2)/2} \|P_{k_1}\partial_\theta u\|_{L_t^4 L_x^{\frac{4}{1+\varepsilon}} L_\theta^3} \|P_{k_2}v\|_{L_t^4 L_x^{\frac{4}{1+\varepsilon}} L_\theta^\infty} \\ & \lesssim 2^{\varepsilon k_3} 2^{-\varepsilon(k_1+k_2)/2} \|P_{k_1}u\|_{\tilde{Y}_{k_1}} \|P_{k_2}v\|_{\tilde{Y}_{k_2}}. \end{aligned}$$

If  $k_1 \leq \min(k_2, k_3)$ , then

$$\begin{aligned} \|P_{k_3}(P_{k_1}\partial_\theta uP_{k_2}v)\|_{L_{t,x}^2} & \lesssim \|P_{k_1}\partial_\theta u\|_{L_t^4 L_x^{\frac{4}{1+\varepsilon}} L_\theta^3} \|P_{k_2}v\|_{L_t^4 L_x^{\frac{4}{1+\varepsilon}} L_\theta^\infty} \\ & \lesssim 2^{\varepsilon(k_1-k_2)/2} \|P_{k_1}u\|_{\tilde{Y}_{k_1}} \|P_{k_2}v\|_{\tilde{Y}_{k_2}}. \end{aligned}$$

If  $k_2 \leq \min(k_1, k_3)$ , the proof is identical to the above case.  $\square$

**Lemma 5.6** (Algebra properties). *If  $s \geq 1$ , then we have*

$$\|uv\|_{Y^s} \lesssim \|u\|_{Y^s} \|v\|_{Y^1} + \|u\|_{Y^1} \|v\|_{Y^s}.$$

*Proof.* By the definition we have  $\|u\|_{L_{x,t}^\infty} \leq \|u\|_{L_t^\infty \dot{B}_{2,1}^1} + \|\partial_\theta u\|_{L_t^\infty \dot{B}_{2,1}^1} \lesssim \|u\|_{Y^1}$ . The Lebesgue component can be easily handled by para-product decomposition and Hölder's inequality. Now we deal with  $X^{s,b}$ -type space. It suffices to show

$$\sum_k \|\partial_\theta^j P_k(fg)\|_{X^{0,1} + \bar{X}^{0,1}} \lesssim \|f\|_{Y^1} \|g\|_{Y^1}, \quad j = 0, 1. \quad (5.5)$$

For simplicity of notations, we write  $X = X^{0,1}$ ,  $\bar{X} = \bar{X}^{0,1}$ . First we consider  $j = 0$ . The left-hand side of the above inequality is bounded by

$$\begin{aligned} \sum_{k_3} \|P_{k_3}(fg)\|_{X+\bar{X}} & \lesssim \sum_{k_i} \|P_k(P_{k_1}fP_{k_2}g)\|_{X+\bar{X}} \\ & \leq \left( \sum_{k_i: k_1 \leq k_2} + \sum_{k_i: k_1 > k_2} \right) \|P_{k_3}(P_{k_1}fP_{k_2}g)\|_{X+\bar{X}} \\ & := I + II. \end{aligned}$$

By symmetry, we only estimate the term  $I$ . Assume  $P_{k_1}f = P_{k_1}f_1 + P_{k_1}f_2$ ,  $P_{k_2}g = P_{k_2}g_1 + P_{k_2}g_2$  such that

$$\|P_{k_1}f_1\|_X + \|P_{k_1}f_2\|_{\bar{X}} \lesssim \|P_{k_1}f\|_{X+\bar{X}}, \quad \|P_{k_2}g_1\|_X + \|P_{k_2}g_2\|_{\bar{X}} \lesssim \|P_{k_2}g\|_{X+\bar{X}}.$$

Then we have

$$\begin{aligned} I &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 \|P_{k_3}(P_{k_1}f_j P_{k_2}g_1)\|_X + \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 \|P_{k_3}(P_{k_1}f_j P_{k_2}g_2)\|_{\bar{X}} \\ &:= I_1 + I_2. \end{aligned}$$

We only estimate the term  $I_1$  since the other term  $I_2$  can be estimated in a similar way. First we assume  $k_3 \leq k_1 + 5$ . We have

$$\begin{aligned} I_1 &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 (\|P_{k_3}Q_{\leq k_1+k_2+9}(P_{k_1}f_j P_{k_2}g_1)\|_X + \|P_{k_3}Q_{\geq k_1+k_2+10}(P_{k_1}f_j P_{k_2}g_1)\|_X) \\ &:= I_{11} + I_{12}. \end{aligned}$$

For the term  $I_{11}$ , by Lemma 5.5 we get

$$\begin{aligned} I_{11} &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_1+k_2} \|P_{k_3}(P_{k_1}f_j P_{k_2}g_1)\|_{L_{t,x}^2} \\ &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_1+k_2} 2^{\varepsilon(k_3-k_1)} \|P_{k_1}f_j\|_{\tilde{Y}_{k_1}} \|P_{k_2}g_1\|_{\tilde{Y}_{k_2}} \\ &\lesssim \|f\|_{Y^1} \|g\|_{Y^1}. \end{aligned}$$

For the term  $I_{12}$ , we need to exploit the nonlinear interactions. We have

$$\begin{aligned} &\mathcal{F}P_{k_3}Q_{\geq k_1+k_2+10}(P_{k_1}f_j P_{k_2}g_1) \\ &= \chi_{k_3}(\xi_3) \chi_{\geq k_1+k_2+10}(\tau_3 + |\xi_3|^2) \int_{\xi_3=\xi_1+\xi_2, \tau_3=\tau_1+\tau_2} \chi_{k_1}(\xi_1) \hat{f}_j(\tau_1, \xi_1) \chi_{k_2}(\xi_2) \hat{g}_1(\tau_2, \xi_2). \end{aligned}$$

We assume  $j = 1$  since  $j = 2$  is similar. On the plane  $\{\xi_3 = \xi_1 + \xi_2, \tau_3 = \tau_1 + \tau_2\}$  we have

$$\tau_3 + |\xi_3|^2 = \tau_1 + |\xi_1|^2 + \tau_2 + |\xi_2|^2 - H(\xi_1, \xi_2) \quad (5.6)$$

where  $H$  is the resonance function in the product  $P_{k_3}(P_{k_1}f_j P_{k_2}g_1)$

$$H(\xi_1, \xi_2) = |\xi_1|^2 + |\xi_2|^2 - |\xi_1 + \xi_2|^2. \quad (5.7)$$

Since  $|H| \lesssim 2^{k_1+k_2}$ , then one of  $P_{k_1}f_j$ ,  $P_{k_2}g_1$  has modulation larger than the output modulation, namely

$$\max(|\tau_1 + |\xi_1|^2|, |\tau_2 + |\xi_2|^2|) \gtrsim |\tau_3 + |\xi_3|^2|.$$

If  $P_{k_1}f_j$  has larger modulation, then

$$\begin{aligned} I_{12} &\lesssim \sum_{k_i: k_1 \leq k_2} \left\| 2^{j_3} \|P_{k_3}Q_{j_3}(P_{k_1}f_j P_{k_2}g_1)\|_{L_{t,x}^2} \right\|_{j_3 \geq k_1+k_2}^2 \\ &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_3} \left( \sum_{j_3 \geq k_1+k_2} 2^{2j_3} \|Q_{\geq j_3}P_{k_1}f_j\|_{L_{t,x}^2}^2 \right)^{1/2} \|P_{k_2}g_1\|_{L_t^\infty L_x^2} \\ &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_3} 2^{k_1} \|P_{k_1}f_j\|_{X+\bar{X}} \|P_{k_2}g_1\|_{Y_{k_2}} \lesssim \|f\|_{Y^1} \|g\|_{Y^1}. \end{aligned}$$

If  $P_{k_1}g_1$  has larger modulation, then

$$\begin{aligned} I_{12} &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_3} \|P_{k_1}f_j\|_{L_t^\infty L_x^2} \left( \sum_{j_3 \geq k_1+k_2} 2^{2j_3} \|P_{k_2}Q_{\geq j_3}g_1\|_{L_t^2 L_x^2}^2 \right)^{1/2} \\ &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_3} 2^{k_2} \|P_{k_1}f_j\|_{Y_{k_1}} \|P_{k_2}g_1\|_X \lesssim \|f\|_{Y^1} \|g\|_{Y^1}. \end{aligned}$$

Now we assume  $k_3 \geq k_1 + 6$ . In this case we have  $|k_2 - k_3| \leq 4$ .

$$\begin{aligned} I_1 &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 (\|P_{k_3}Q_{\leq k_1+k_2+9}(P_{k_1}f_j P_{k_2}g_1)\|_X + \|P_{k_3}Q_{\geq k_1+k_2+10}(P_{k_1}f_j P_{k_2}g_1)\|_X) \\ &:= \tilde{I}_{11} + \tilde{I}_{12}. \end{aligned}$$

By Lemma 5.5 we get

$$\begin{aligned} \tilde{I}_{11} &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 (\|P_{k_3}Q_{\leq k_1+k_2+9}(P_{k_1}f_j P_{k_2}g_1)\|_X \\ &\lesssim \sum_{k_i: k_1 \leq k_2} 2^{k_1+k_2} \|P_{k_1}f_j\|_{L_{t,x}^4} \|P_{k_2}g_1\|_{L_{t,x}^4} \lesssim \|f\|_{Y^1} \|g\|_{Y^1}. \end{aligned}$$

For the term  $\tilde{I}_{12}$ , similarly as the term  $I_{12}$ , we may assume  $P_{k_1}f_j$  (or  $P_{k_2}g_1$ ) has modulation  $\gtrsim 2^{k_1+k_2}$ . If  $P_{k_1}f_j$  has larger modulation, then

$$\begin{aligned} I_{12} &\lesssim \sum_{k_i: k_1 \leq k_2} \left\| 2^{j_3} \|P_{k_3}Q_{j_3}(P_{k_1}f_j P_{k_2}g_1)\|_{L_{t,x}^2} \right\|_{l_{j_3 \geq k_1+k_2}^2} \\ &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_1} \left( \sum_{j_3 \geq k_1+k_2} 2^{2j_3} \|Q_{\geq j_3}P_{k_1}f_j\|_{L_{t,x}^2}^2 \right)^{1/2} \|P_{k_2}g_1\|_{L_t^\infty L_x^2} \\ &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_1} 2^{k_1} \|P_{k_1}f_j\|_{X+\bar{X}} \|P_{k_2}g_1\|_{Y_{k_2}} \lesssim \|f\|_{Y^1} \|g\|_{Y^1}. \end{aligned}$$

If  $P_{k_1}g_1$  has larger modulation, then

$$\begin{aligned} I_{12} &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_1} \|P_{k_1}f_j\|_{L_t^\infty L_x^2} \left( \sum_{j_3 \geq k_1+k_2} 2^{2j_3} \|P_{k_2}Q_{\geq j_3}g_1\|_{L_t^2 L_x^2}^2 \right)^{1/2} \\ &\lesssim \sum_{k_i: k_1 \leq k_2} \sum_{j=1}^2 2^{k_1} 2^{k_2} \|P_{k_1}f_j\|_{Y_{k_1}} \|P_{k_2}g_1\|_X \lesssim \|f\|_{Y^1} \|g\|_{Y^1}. \end{aligned}$$

For  $j = 1$  in (5.5), we see the above argument can be easily modified. Thus, we complete the proof.  $\square$

**Lemma 5.7.** *We have*

$$\begin{aligned} &\sum_{k_1, k_2, k_3} (\|P_{k_3}[u \sum_{i=1}^2 (\partial_{x_i} P_{k_1} v \partial_{x_i} P_{k_2} w)]\|_{L_{t,x}^2} + \|\partial_\theta P_{k_3}[u \sum_{i=1}^2 (\partial_{x_i} P_{k_1} v \partial_{x_i} P_{k_2} w)]\|_{L_{t,x}^2}) \\ &\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}. \end{aligned} \tag{5.8}$$

*Proof.* We only estimates the first term on the left-hand side, since the other term is similar. We have

$$\begin{aligned} \text{First term on LHS of (5.8)} &\lesssim \sum_{k_1, k_2, k_3} \|P_{k_3}[P_{\geq k_3-10} u \sum_{i=1}^n (\partial_{x_i} P_{k_1} v \partial_{x_i} P_{k_2} w)]\|_{L_{t,x}^2} \\ &\quad + \sum_{k_1, k_2, k_3} \|P_{k_3}[P_{\leq k_3-10} u \sum_{i=1}^n (\partial_{x_i} P_{k_1} v \partial_{x_i} P_{k_2} w)]\|_{L_{t,x}^2} \\ &:= I + II. \end{aligned}$$

For the term  $I$ , by Lemma 4.2 we get

$$\begin{aligned} I &\lesssim \sum_{k_1, k_2, k_3} 2^{k_3} \|P_{\geq k_3-10} u\|_{L_t^\infty L_x^2} \left\| \sum_{i=1}^n (\partial_{x_i} P_{k_1} v \partial_{x_i} P_{k_2} w) \right\|_{L_{t,x}^2} \\ &\lesssim \sum_{k_1, k_2, k_3} 2^{k_3} 2^{k_1+k_2} \|P_{\geq k_3-10} u\|_{L_t^\infty L_x^2} \|P_{k_1} v\|_{F_{k_1}} \|P_{k_2} w\|_{F_{k_2}} \\ &\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}. \end{aligned}$$

For the term  $II$  we have

$$II \lesssim \|u\|_{Y^1} \sum_{k_1, k_2, k_3} \|\tilde{P}_{k_3} \sum_{i=1}^n (\partial_{x_i} P_{k_1} v \partial_{x_i} P_{k_2} w)\|_{L_{t,x}^2}$$

We may assume  $k_1 \leq k_2$  in the above summation. If  $k_3 \geq k_2 - 9$ , then  $|k_3 - k_2| \leq 5$  and thus

$$\begin{aligned} II &\lesssim \|u\|_{Y^1} \sum_{k_1 \leq k_2, |k_2 - k_3| \leq 5} 2^{k_1+k_2} 2^{(k_1-k_2)/2} \|P_{k_1} v\|_{F_{k_1}} \|P_{k_2} w\|_{F_{k_2}} \\ &\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}. \end{aligned}$$

If  $k_3 \leq k_2 - 10$ , then  $|k_1 - k_2| \leq 5$ . Thus we get

$$\begin{aligned} II &\lesssim \|u\|_{Y^1} \sum_{k_1, k_2, k_3} 2^{k_3 \varepsilon} \|\tilde{P}_{k_3} \sum_{i=1}^n (\partial_{x_i} P_{k_1} v \partial_{x_i} P_{k_2} w)\|_{L_t^2 L_x^{\frac{2}{1+\varepsilon}}} \\ &\lesssim \|u\|_{Y^1} \sum_{k_1, k_2, k_3} 2^{k_3 \varepsilon} 2^{k_1+k_2} \|P_{k_1} v \partial_{x_i}\|_{L_t^2 L_x^{\frac{4}{1+\varepsilon}} L_\theta^3} \|P_{k_2} w\|_{L_t^2 L_x^{\frac{4}{1+\varepsilon}} L_\theta^\infty} \\ &\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}. \end{aligned}$$

Therefore we complete the proof.  $\square$

**Lemma 5.8.** *We have*

$$\|u \sum_{i=1}^2 (\partial_{x_i} v \partial_{x_i} w)\|_{W^1} \lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}. \quad (5.9)$$

*Proof.* By the definition, we have

$$\|u \sum_{i=1}^2 \partial_{x_i} v \partial_{x_i} w\|_{W^1} \leq \sum_{k_i} 2^{k_4} \sum_{j=0,1} \|\partial_\theta^j P_{k_4} [P_{k_1} u \sum_{i=1}^2 (P_{k_2} \partial_{x_i} v \partial_{x_i} P_{k_3} w)]\|_{W_{k_4}}. \quad (5.10)$$

The  $L_{t,x}^2$  component in  $W_{k_4}$  is handled by the previous lemma. So we only need to handle the other component. We assume  $j = 0$  in the above inequality since the

other case  $j = 1$  is similar. By symmetry we may assume  $k_2 \leq k_3$  in the above summation. If in the above summation we assume  $k_4 \leq k_1 + 40$ , then

$$\begin{aligned}
(5.10) &\lesssim \sum_{k_i} 2^{k_4} \|P_{k_4} [P_{k_1} u P_{k_2} \partial_{x_i} \bar{v} \partial_{x_i} P_{k_3} w]\|_{L_{t,x}^{4/3}} \\
&\lesssim \sum_{k_i} 2^{k_1} \|P_{k_1} u\|_{L_{x,t}^4} 2^{k_2} \|P_{k_2} v\|_{L_{x,t}^4} 2^{k_3} \|P_{k_3} w\|_{L_{x,t}^4} \\
&\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}.
\end{aligned}$$

Thus we assume  $k_4 \geq k_1 + 40$  in the summation of (5.10). We bound the summation case by case.

**Case 1:**  $k_2 \leq k_1 + 20$

In this case we have  $k_4 \geq k_2 + 20$  and hence  $|k_4 - k_3| \leq 5$ . By Lemma 4.2 we get

$$\begin{aligned}
(5.10) &\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} \|P_{k_4} [P_{k_1} u P_{k_2} \partial_{x_i} \bar{v} \partial_{x_i} P_{k_3} w]\|_{L_e^{1,2}} \\
&\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} \|P_{k_1} u P_{k_3} \partial_{x_i} w\|_{L_{x,t}^2} \|P_{k_2} \partial_{x_i} v\|_{L_e^{2,\infty}} \\
&\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}.
\end{aligned}$$

**Case 2:**  $k_2 \geq k_1 + 21$

In this case we have  $k_4 \leq k_3 + 40$ . Let  $g = \sum_{i=1}^n (P_{k_2} \partial_{x_i} v \cdot P_{k_3} \partial_{x_i} w)$ . Then we have

$$\begin{aligned}
(5.10) &\lesssim \sum_{k_i} 2^{k_4} \|P_{k_4} [P_{k_1} u Q_{\leq k_2+k_3} g]\|_{N_{k_4}} + \sum_{k_i} 2^{k_4} \|P_{k_4} [P_{k_1} u Q_{\geq k_2+k_3} g]\|_{N_{k_4}} \\
&:= I + II.
\end{aligned}$$

First we estimate the term  $II$ . We have

$$\begin{aligned}
II &\lesssim \sum_{k_i} 2^{k_4} \|P_{k_4} [P_{k_1} Q_{\geq k_2+k_3-10} u \cdot Q_{\geq k_2+k_3} g]\|_{N_{k_4}} \\
&\quad + \sum_{k_i} 2^{k_4} \|P_{k_4} [P_{k_1} Q_{\leq k_2+k_3-10} u \cdot Q_{\geq k_2+k_3} g]\|_{N_{k_4}} \\
&:= II_1 + II_2.
\end{aligned}$$

For the term  $II_1$  we have

$$\begin{aligned}
II_1 &\lesssim \sum_{k_i} 2^{k_4} \|P_{k_4} [P_{k_1} Q_{\geq k_2+k_3-10} u \cdot Q_{\geq k_2+k_3} g]\|_{L_t^1 L_x^2} \\
&\lesssim \sum_{k_i} 2^{k_4} \|P_{k_1} Q_{\geq k_2+k_3-10} u\|_{L_t^2 L_x^\infty} \|Q_{\geq k_2+k_3} g\|_{L_t^2 L_x^2} \\
&\lesssim \sum_{k_i} 2^{k_4} 2^{k_1} \|P_{k_1} Q_{\geq k_2+k_3-10} u\|_{L_t^2 L_x^2} \|Q_{\geq k_2+k_3} g\|_{L_t^2 L_x^2} \\
&\lesssim \sum_{k_i} 2^{k_4} 2^{k_1} 2^{-(k_2+k_3)} \|P_{k_1} u\|_{X^{0,1} + \bar{X}^{0,1}} 2^{(k_2-k_3)/2} 2^{k_2+k_3} \|P_{k_2} v\|_{F_{k_2}} \|P_{k_3} w\|_{F_{k_3}} \\
&\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}.
\end{aligned}$$

For the term  $II_2$ , since  $k_4 \geq k_1 + 40$ , then we may assume  $g$  has frequency of size  $2^{k_4}$ . The resonance function in the product  $P_{k_1} u \cdot p_{k_4} g$  is of size  $\lesssim 2^{k_1+k_4}$ . Thus the

output modulation is of size  $\gtrsim 2^{k_2+k_3}$ . Then we get

$$\begin{aligned}
II_2 &\lesssim \sum_{k_i} 2^{k_4} 2^{-(k_2+k_3)/2} \|P_{k_4}[P_{k_1}Q_{\leq k_2+k_3-10}u \cdot Q_{\geq k_2+k_3}g]\|_{L_{t,x}^2} \\
&\lesssim \sum_{k_i} 2^{k_4} 2^{-(k_2+k_3)/2} 2^{k_1} \|P_{k_1}u\|_{L_t^\infty L_x^2} \cdot \|g\|_{L_{t,x}^2} \\
&\lesssim \sum_{k_i} 2^{k_4} 2^{-(k_2+k_3)/2} 2^{k_1} 2^{(k_2-k_3)/2} 2^{k_2+k_3} \|P_{k_1}u\|_{Y_{k_1}} \|P_{k_2}v\|_{F_{k_2}} \|P_{k_3}w\|_{F_{k_3}} \\
&\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}.
\end{aligned}$$

Now we estimate the term  $I$ . We have

$$\begin{aligned}
I &\lesssim \sum_{k_i} 2^{k_4} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_2+k_3} \sum_{i=1}^2 (P_{k_2}\partial_{x_i}Q_{\geq k_2+k_3+40}v \cdot P_{k_3}\partial_{x_i}w)]\|_{N_{k_4}} \\
&\quad + \sum_{k_i} 2^{k_4} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_2+k_3} \sum_{i=1}^2 (P_{k_2}\partial_{x_i}Q_{\leq k_2+k_3+39}v \cdot P_{k_3}\partial_{x_i}w)]\|_{N_{k_4}} \\
&:= I_1 + I_2.
\end{aligned}$$

For the term  $I_1$ , since the resonance function in the product  $P_{k_2}v \cdot p_{k_3}w$  is of size  $\lesssim 2^{k_2+k_3}$ , then we may assume  $P_{k_3}w$  has modulation of size  $\gtrsim 2^{k_2+k_3}$ . Then we get

$$\begin{aligned}
I_1 &\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_2+k_3} \sum_{i=1}^2 (P_{k_2}\partial_{x_i}Q_{\geq k_2+k_3+40}v \cdot P_{k_3}\partial_{x_i}Q_{\geq k_2+k_3-5}w)]\|_{L_e^{1,2}} \\
&\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} \|P_{k_1}u\|_{L_{t,x}^\infty} 2^{k_2+k_3} \|P_{k_2}v\|_{L_e^{2,\infty}} \|P_{k_3}Q_{\geq k_2+k_3-5}w\|_{L_{t,x}^2} \\
&\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} 2^{(k_2+k_3)/2} 2^{k_2/2} 2^{k_1} \|P_{k_1}u\|_{Y_{k_1}} \|P_{k_2}v\|_{F_{k_2}} \|P_{k_3}w\|_{F_{k_3}} \\
&\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}.
\end{aligned}$$

Finally, we estimate the term  $I_2$ . For this term, we need to use the null structure observed by Bejenaru [2]. We can rewrite

$$2\nabla u \cdot \nabla v = (i\partial_t - \Delta)u \cdot v + u \cdot (i\partial_t - \Delta)v - (i\partial_t - \Delta)(u \cdot v). \quad (5.11)$$

Let  $L = i\partial_t - \Delta$ . Then we have

$$\begin{aligned}
I_2 &= \sum_{k_i} 2^{k_4} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_2+k_3} (P_{k_2}LQ_{\leq k_2+k_3+39}v \cdot P_{k_3}w)]\|_{N_{k_4}} \\
&\quad + \sum_{k_i} 2^{k_4} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_2+k_3} (P_{k_2}Q_{\leq k_2+k_3+39}v \cdot P_{k_3}Lw)]\|_{N_{k_4}} \\
&\quad + \sum_{k_i} 2^{k_4} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_2+k_3} L(P_{k_2}Q_{\leq k_2+k_3+39}v \cdot P_{k_3}w)]\|_{N_{k_4}} \\
&:= I_{21} + I_{22} + I_{23}.
\end{aligned}$$

For the term  $I_{21}$ , if  $k_4 \geq k_2 + 10$ , then  $|k_4 - k_3| \leq 5$  and hence

$$\begin{aligned} I_{21} &\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_2+k_3}(P_{k_2}LQ_{\leq k_2+k_3+39}v \cdot P_{k_3}w)]\|_{L_e^{1,2}} \\ &\lesssim \sum_{k_i} 2^{k_4} 2^{k_1} 2^{-k_4/2} \|P_{k_1}u\|_{L_t^\infty L_x^2} \|P_{k_2}Lv\|_{L_{t,x}^2} \|P_{k_3}w\|_{L_e^{2,\infty}} \\ &\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}. \end{aligned}$$

On the other hand, if  $k_4 \leq k_2 + 10$ , we get

$$\begin{aligned} I_{21} &\lesssim \sum_{k_i} 2^{k_4} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_2+k_3}(P_{k_2}LQ_{\leq k_2+k_3+39}v \cdot P_{k_3}w)]\|_{L_{t,x}^{4/3}} \\ &\lesssim \sum_{k_i} 2^{k_4} 2^{k_1} \|P_{k_1}u\|_{L_t^\infty L_x^2} \|P_{k_2}Lv\|_{L_{t,x}^2} \|P_{k_3}w\|_{L_{t,x}^4} \\ &\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}. \end{aligned}$$

For the term  $I_{22}$ , we may assume  $w$  has modulation  $\lesssim 2^{k_2+k_3}$ . Then we get

$$\begin{aligned} I_{22} &\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_2+k_3}(P_{k_2}Q_{\leq k_2+k_3+39}v \cdot P_{k_3}Q_{\leq k_2+k_3+100}Lw)]\|_{L_e^{1,2}} \\ &\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} 2^{k_1} \|P_{k_1}u\|_{L_t^\infty L_x^2} \|P_{k_2}v\|_{L_e^{2,\infty}} \|P_{k_3}Q_{\leq k_2+k_3+100}Lw\|_{L_{t,x}^2} \\ &\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} 2^{k_1} 2^{k_2/2} 2^{(k_2+k_3)/2} \|P_{k_1}u\|_{Y_{k_1}} \|P_{k_2}v\|_{F_{k_2}} \|P_{k_3}w\|_{X^{0,1/2,\infty}} \\ &\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}. \end{aligned}$$

Next we estimate the term  $I_{23}$ . We have

$$\begin{aligned} I_{23} &\lesssim \sum_{k_i} 2^{k_4} \|P_{k_4}[P_{k_1}u \cdot Q_{[k_1+k_4+100, k_2+k_3]}L(P_{k_2}Q_{\leq k_2+k_3+39}v \cdot P_{k_3}w)]\|_{N_{k_4}} \\ &\quad + \sum_{k_i} 2^{k_4} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_1+k_4+99}L(P_{k_2}Q_{\leq k_2+k_3+39}v \cdot P_{k_3}w)]\|_{N_{k_4}} \\ &:= I_{231} + I_{232}. \end{aligned}$$

For the term  $I_{232}$  we have

$$\begin{aligned} I_{232} &\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} \|P_{k_4}[P_{k_1}u \cdot Q_{\leq k_1+k_4+99}L(P_{k_2}Q_{\leq k_2+k_3+39}v \cdot P_{k_3}w)]\|_{L_e^{1,2}} \\ &\lesssim \sum_{k_i} 2^{k_4} 2^{-k_4/2} \|P_{k_1}u\|_{L_e^{2,\infty}} 2^{k_1+k_4} 2^{(k_2-k_3)/2} \|P_{k_2}v\|_{F_{k_2}} \|P_{k_3}w\|_{F_{k_3}} \\ &\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}. \end{aligned}$$

For the term  $I_{231}$  we have

$$\begin{aligned} I_{231} &\lesssim \sum_{k_i} \sum_{j_2=k_1+k_4+100}^{k_2+k_3} 2^{k_4} \|P_{k_4}Q_{\leq j_2-10}[P_{k_1}u \cdot Q_{j_2}L(P_{k_2}Q_{\leq k_2+k_3+39}v \cdot P_{k_3}w)]\|_{N_{k_4}} \\ &\quad + \sum_{k_i} \sum_{j_2=k_1+k_4+100}^{k_2+k_3} 2^{k_4} \|P_{k_4}Q_{\geq j_2-9}[P_{k_1}u \cdot Q_{j_2}L(P_{k_2}Q_{\leq k_2+k_3+39}v \cdot P_{k_3}w)]\|_{N_{k_4}} \\ &:= I_{2311} + I_{2312}. \end{aligned}$$



For the term  $I_{2312}$  we have

$$\begin{aligned}
I_{2312} &\lesssim \sum_{k_i} \sum_{j_2=k_1+k_4+100}^{k_2+k_3} \sum_{j_3 \geq k_2-9} 2^{k_4} 2^{-j_3/2} \|P_{k_4} Q_{j_3} [P_{k_1} u \cdot Q_{j_2} L(P_{k_2} Q_{\leq k_2+k_3+39} v \cdot P_{k_3} w)]\|_{L_{t,x}^2} \\
&\lesssim \sum_{k_i} 2^{k_4} 2^{k_1} 2^{(k_2+k_3)/2} 2^{(k_2-k_3)/2} \|P_{k_1} u\|_{Y_{k_1}} \|P_{k_2} v\|_{F_{k_2}} \|P_{k_3} w\|_{F_{k_3}} \\
&\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}.
\end{aligned}$$

For the term  $I_{2311}$  we have

$$\begin{aligned}
I_{2311} &\lesssim \sum_{k_i} \sum_{j_2=k_1+k_4+100}^{k_2+k_3} 2^{k_4} \|P_{k_4} Q_{\leq j_2-10} [P_{k_1} \tilde{Q}_{j_2} u \cdot Q_{j_2} L(P_{k_2} Q_{\leq k_2+k_3+39} v \cdot P_{k_3} w)]\|_{L_t^1 L_x^2} \\
&\lesssim \sum_{k_i} \sum_{j_2=k_1+k_4+100}^{k_2+k_3} 2^{k_4} 2^{k_1} \|P_{k_1} \tilde{Q}_{j_2} u\|_{L_{t,x}^2} 2^{j_2} 2^{(k_2-k_3)/2} \|P_{k_2} v\|_{F_{k_2}} \|P_{k_3} w\|_{F_{k_3}} \\
&\lesssim \|u\|_{Y^1} \|v\|_{Z^1} \|w\|_{Z^1}.
\end{aligned}$$

Therefore, we complete the proof.  $\square$

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